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# PLANE AND SOLID GEOMETRY

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# PREFACE

Greek Geometry, the finest product of deductive thinking which high school pupils encounter, has come down to us through twenty-two centuries practically unchanged in essential content or form. It has been presented in texts, each built upon a preceding one and each good in its day, which have sought to present the great science in accord with the ideals of their time.

This text is a thorough revision of Wells's Essentials of Geometry in accord with current scientific and pedagogical thought. The scientific ideal is represented the world over by Hilbert's *Foundations of Geometry*. (Translated by Townsend, Open Court Pub. Co., Chicago.) The pedagogical ideals are represented in this country by the Report of the National Committee of Fifteen. (See *Mathematics Teacher*, Dec., 1912; *School Science and Mathematics*, 1911; *Proceedings of N. E. A.*, 1911.) These ideals and the personal experience of one of the authors in teaching high school geometry in recent years have been the determining factors in the making of this text. Permit us to direct attention to some of its features.

In each Book, the fundamentally important theorems are given first. These theorems present a *safe and sāne minimum course.* These are followed in each Book by one or more groups of theorems or applications which are strictly supplementary, — material which either has long appeared in geometries in some form or has been introduced in recent years to add to the pupils' interest. Teachers will find no difficulty in

making selection from this material, and, on the other hand, will not be embarrassed by omitting any of it. (See pp. 172, 210, and 245.)

The introduction presents only the immediately necessary concepts, notation, and terminology. Emphasis is upon the acquisition of these and of skill in the use of tools, and above all upon the acquisition of the important point of view presented in §§ 48–50.

The fundamental constructions are placed early in Book I so that pupils can be required to construct their figures; they are not placed earlier because they cannot be proved earlier.

Authorities and details of demonstrations which pupils can supply are increasingly omitted from the demonstrations, and often only suggestions are given. The resulting proofs are an incentive to real thinking for all the members of the class; they do not consume time that can be spent more profitably upon exercises and other valuable supplementary material.

Pupils are encouraged to plan their proofs instead of plunging blindly into a demonstration. (See §§ 69 and 117.)

Unnecessary corollaries have been omitted, and dignity and importance is given to those which are included in the text. (See §§ 71, 96, 101.)

The stages of the proof are plainly marked, the steps are numbered, the reasons are given in full, and the proofs are arranged attractively on the page.

Carefully selected exercises follow most of the propositions. Notice exercises such as Exercises 2, 23, 45, 63, of Book I, designed to teach concretely and inductively the theorems which immediately follow. Notice also the illustrative exercises which set a standard for the pupils' solution. (See pp. 31, 32, 157.) Enough exercises are provided for a minimum course. Besides these, there are miscellaneous exercises at the close of each Book, depending upon only the theorems of the minimum course. Finally there will be found from time to time a note like that on page 52, referring to supplementary exercises at the end of the text. (See pp. 52, 59, 83.) Suggestions are

given with exercises where experience has shown that a majority of a class require such assistance in order to do effective work. (See Book I, Ex. 128, 131, etc.)

Simple applied problems (see Book I, Ex. 15, 37, 39, 40, 41, etc.) and artistic designs (see pp. 1, 47, 50, etc.) exhibit to the pupils some of the uses of geometry. Only simple applications are included in the minimum course. Other applications are introduced among the supplementary exercises at the end of the text and among the supplementary topics at the close of certain of the Books. (See pp. 138, 172, 174, 246, etc.)

A brief history of geometry is included in the introduction, and other historical references are introduced from time to time throughout the text. (See pp. 29, 36, 46, 240, etc.)

Axioms are defined in the accepted modern form (p. 22). They are introduced only as they become necessary. (See pp. 22, 29, 50, 82.) In the introduction, their meaning is made clear by suitable preliminary exercises. (See Introduction, Ex. 22, 23, 24, 37, etc.) The definitions also are modern and consistent, even though they are in some cases different from those ordinarily given. (See §§ 1, 2, 4, 5, 47, the note on p. 27, etc.) For example, after defining a circle as a line, which is correct, there is every reason for also defining a polygon as a line, instead of defining it as a portion of a plane. It may seem strange at first also not to find in the first paragraph of the text the attempted distinction between a physical and a geometrical solid, — something that is psychologically impossible for beginners, — but the authors believe firmly that there is much to recommend their own informal statements in § 1.

The incommensurable cases are dismissed with a mere remark on pages 113, 149, and 194, and are treated fully only after the theory of limits is given on page 260.

The mensuration of the circle is treated informally at first on page 238. The treatment involves nevertheless the basic ideas which are developed more fully in the formal treatment of the same topic which appears as one of the supplementary topics of Book V on page 248. This treatment is as elemen-

tary as the difficulty of the subject permits; to give less would render the treatment either incomprehensible or incomplete. On the other hand, the treatment is as sound as an elementary presentation renders possible; to give more would certainly render the subject distasteful to an average high school class.

In the treatment of the mensuration of the cylinder and the cone, the fundamental limits theorems are assumed on the ground that rigorous proofs are beyond the scope of an elementary course. In the enunciation of the area theorems for portions of the surface of a sphere, changes have been made which enable pupils both to learn and to remember the theorems more readily.

The course in Solid Geometry is *practical* in the sense that the mensuration theorems for the common solids are given the place of prominence. For example, in Book IX the mensuration of the sphere is treated in the minimum course, — the mathematically interesting theorems about spherical geometry being grouped as a supplementary topic. Besides this emphasis given to the mensuration theorems, some natural applications of solid geometry are touched upon in the exercises.

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PLANE AND SOLID GEOMETRY

## SYMBOLS

| | |
|---|---|
| $=$, is equal to; equals. | $\angle$, angle. |
| $>$, is greater than. | $\triangle$, triangle. |
| $<$, is less than. | $\square$, parallelogram. |
| $\parallel$, is parallel to; parallel. | $\sqsubset\!\sqsupset$, rectangle. |
| $\perp$, is perpendicular to. | $\odot$, circle. |
| $\perp$, perpendicular. | $\therefore$, therefore. |
| $\sim$, is similar to. | $\equiv$, is identically equal to. |
| $\cong$, is congruent to. | $\doteq$, approaches as limit. |

Any symbol representing a noun is converted into the plural by affixing the letter *s*; thus $\angle s$ means angles.

## ABBREVIATIONS

| | | | |
|---|---|---|---|
| *Adj.*, | adjacent. | *Ex.*, | exercise. |
| *Alt.*, | alternate. | *Ext.*, | exterior. |
| *Ax.*, | axiom. | *Hom.*, | homologous. |
| *Con.*, | conclusion. | *Hyp.*, | hypothesis. |
| *Cong.*, | congruent. | *Int.*, | interior. |
| *Const.*, | construction. | *Rect.*, | rectangle. |
| *Cor.*, | corollary. | *Rt.*, | right. |
| *Corres.*, | corresponding. | *St.*, | straight. |
| *Def.*, | definition. | *Supp.*, | supplementary. |

It is not necessary to learn any of these symbols until they are introduced in the text.

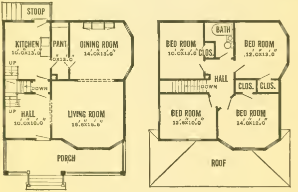
A View in the Congressional Library, Illustrating the Use of Geometry in Architecture

# GEOMETRY

## INTRODUCTION

In arithmetic and algebra, frequent reference is made to the rectangle, the square, the triangle, and the circle. These are geometrical figures, and in geometry a careful study of them and of many others is made.

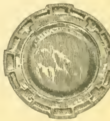
Geometrical figures are used constantly in architecture.




Plans of a House

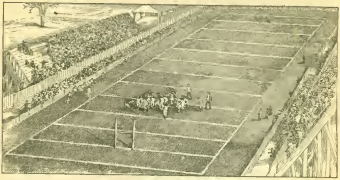
They often form the basis of artistic designs.

A Textile Pattern



An Artistic Tray

Our playgrounds are often laid out in geometrical forms.



A FOOTBALL FIELD

Familiarity with such figures and their properties, and ability to construct and measure them, is both interesting and worth while. It is interesting also to know how man has developed his knowledge of such figures and his skill in using them.

## HISTORY OF GEOMETRY

Geometry as it is now studied has been handed down to us from the Greeks. The word "geometry" is derived from two Greek words meaning *the earth* and *to measure;* this fact is evidence that the Greeks believed that geometry was intimately associated with or else had been developed out of the practical business of measuring the earth, — surveying.

The Greeks received their start in geometry from the Egyptians. Thales of Miletus (630–550 B.C.) is given special credit for transplanting a knowledge of Egyptian geometry to Greece.

Did the Egyptians originate geometry? Whether they did or not, there is evidence that they had some knowledge of practical geometry. Their pyramids and other marvelous structures point to this fact. Also, there is in the British Museum a papyrus written about 1700 B.C. by an Egyptian, commonly

called Ahmes, which contains among other interesting mathematical records some formulæ for measuring geometrical figures. This papyrus is a copy of another written before the time of Ahmes. Herodotus, a Greek traveler and historian, is said to be responsible for the story that the Egyptians developed these rules of mensuration because of the necessity of frequently surveying the lands which were inundated by the floods of the Nile. The Egyptians must have obtained their formulæ by experiment or by observation. Some of the formulæ were incorrect and their formula for measuring the area of a circle was less accurate than that developed later by the Greeks.

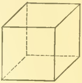
The Greeks became interested in geometry for its own sake as well as for its usefulness. In the three hundred years following the time of Thales, geometry grew into a great science in their schools, far exceeding the geometry of the Egyptians in the number and interest of the facts discovered, and in the accuracy and usefulness of the results. Pythagoras and Plato were the leaders of two groups of students which were responsible for much of the advance made in the subject.

Hippocrates (about 420 B.C.) made an attempt to prepare a text on geometry, but it remained for Euclid to write what became the standard text. Euclid lived between 330 and 275 B.C. He was one of the first and greatest mathematicians who taught at the University of Alexandria. As a teacher he felt the need of a text by which to lead beginners through the known facts of elementary geometry. He therefore gathered together and systematized these facts in a book known as the Elements. Euclid's *Elements* has stood as the model for all subsequent texts on the subject.

During the two thousand years since the time of Euclid, geometry has been studied by all civilized peoples and has been enriched from time to time by their mathematicians. This history is so long and the details are so technical that it is unwise to attempt to give more of it at this time.

### INFORMAL PREPARATORY GEOMETRY

**1.** The adjoining figure is a *cube*. It has six surfaces. Each surface is bounded by four lines, — straight lines. Each straight line is bounded by two points.



The surfaces of a cube, which are smooth and flat, are called *Plane Surfaces;* they are such that a straightedge (ruler) will touch the surface at all points of the straightedge, no matter where the plane surface may be tested.

**2. Plane Geometry** is the study of figures like the square, the triangle, the circle, etc., —figures which lie in a plane surface.

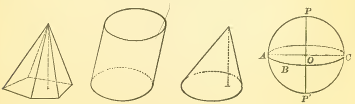
A **Plane Geometrical Figure** is a combination of points and lines which lie in one plane surface. Only such figures are considered in plane geometry.

**Ex. 1.** Test the surface of your desk with your ruler to determine whether the surface is a plane or not. (See § 1.)

**Ex. 2.** What are some other objects which have plane surfaces?

**Ex. 3.** How do men who are laying a concrete walk make use of this test in order to make the surface of the walk approximately plane?

**3. Solid Geometry** is the study of figures like the cube, the sphere, the cylinder, the pyramid, the cone, etc.




PYRAMID CYLINDER CONE SPHERE

**4.** A **Point** is represented to the eye by a small dot. $\cdot A$

A point is named by placing beside it a capital printed letter; as point $A$.

A point represents position only.

**5.** A **Straight Line** is represented to the eye by a mark made by drawing a pencil, a pen, or a piece of crayon along the edge of a straightedge.

A line represents length only.

A **Curved Line** is a line no part of which is straight.

A **Broken Line** is a line composed of different successive straight lines.


STRAIGHT LINE CURVED LINE BROKEN LINE

**6.** Lines like the adjoining ones are called *closed lines.*

It is apparent that a closed line incloses a portion of the plane.

**7.** The word "line" will mean a straight line hereafter unless otherwise specified.

**Ex. 4.** Place upon paper a single point. ($a$) Draw through it one straight line. ($b$) Can you draw through it another straight line? ($c$) A third? ($d$) How many straight lines can be drawn through one point?

**Ex. 5.** Place upon paper a point $A$ and a point $B$. ($a$) Draw from $A$ to $B$ a straight line. ($b$) What happens when you try to draw a second straight line from $A$ to $B$? ($c$) How many different straight lines do you conclude can be drawn between two points?

**Ex. 6.** Can more than one curved line be drawn between two points? Illustrate.

**Ex. 7.** ($a$) When walking along a straight line, are you moving constantly in the same direction or not? ($b$) Answer the same question if you are walking along a curved line.

**Ex. 8.** Draw a straight line 2 inches long. Extend it one inch in each direction.

**8.** It will be assumed as apparent from the preceding exercises that:

(*a*) *One and only one straight line can be drawn through two points.*

This fact is also expressed thus: *two points determine a straight line.*

(*b*) *A straight line can be extended indefinitely in each direction.*

**9.** The straight line determined by points $A$ and $B$ is called the *line AB.*


**Ex. 9.** Select three points which are not in one straight line. Letter them $A$, $B$, and $C$. Draw the different straight lines determined by them taken two at a time. Name the straight lines that you get.

**Ex. 10.** If four towns are situated so that no three can be connected by one straight road, how many roads must be constructed if each town is to be connected with each of the others by a straight road? Illustrate by a drawing.

**Ex. 11.** Draw the straight line determined by two points. Then turn the straightedge over, and again draw a straight line between the two points. If the edge is a true straightedge, the two straight lines will coincide (form one line). Why is this so?

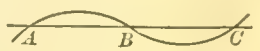
**Ex. 12.** Make a straightedge by folding a piece of paper. Test it by the method suggested in the preceding exercise.

**Ex. 13.** In order to walk across a field in a straight line, a boy selects two objects which are in the direction in which he wishes to go, one of them directly between him and the other. As he walks, he constantly keeps the first object between himself and the second.

(*a*) Why can he guide himself in this manner?

(*b*) What two points determine the straight line along which he walks?

**10.** Two lines, straight or curved, **intersect** if they have one or more common points. The common points are called **Points of Intersection.**



**11.** *Two straight lines can intersect at only one point.*


If they were to intersect in two points, there would be two straight lines through these two points, and this is impossible (§ 8).

This fact is also stated thus: *two intersecting straight lines determine a point.*

**Ex. 14.** Draw three straight lines intersecting by pairs which do not all pass through one point. How many points do they determine?

**Ex. 15.** If there are in a county four straight roads, each of which crosses each of the others, and no three of which meet at one point, how many crossings are there? Illustrate.

**Ex. 16.** How many points are determined by five straight lines intersecting by pairs, no three of which pass through a common point?

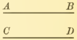
**Ex. 17.** Can you make any definite statement about the number of points of intersection of two curved lines?

**12.** A **Line-segment** or **Segment** is the part of a straight line between two points of the line; as, segment $RS$.


**13.** Two **segments** are **equal** if they can be placed so that the ends of the one are exactly upon the ends of the other.

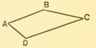
The tool for testing the equality of two segments is the dividers.

The dividers are spread until the points are upon $A$ and $B$ respectively. If the dividers can then be placed with their points on $C$ and $D$ respectively without changing the position of the legs of the dividers, then the two segments are equal.



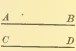

$AB$ is less than $(<)$ $CD$ if $AB$ equals a part of $CD$.

**Ex. 18.** Determine by means of the dividers the relative lengths of $AB$ and $BC$; of $AB$ and $CD$; of $AB$ and $AD$.




**Ex. 19.** Draw any segment $AB$. On a line of indefinite length, mark off from a point $O$ of that line a segment equal to $2\ AB$; also one equal to $3\ AB$.

**Ex. 20.** Draw segments $AB$ and $CD$, with $AB$ greater than $CD$.
(*a*) On a line of indefinite length, mark off a segment equal to $AB + CD$. (*b*) Mark off a segment equal to $AB - CD$.

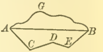



**Ex. 21.** Let $AB$ and $CD$ be two segments. Suppose that $AB$ is placed upon $CD$ with point $A$ on point $C$. (*a*) Where will $B$ fall if $AB = CD$?
(*b*) Where, if $AB = \frac{1}{2}\ CD$?
(*c*) Where, if $AB$ is greater than $CD$?

**Ex. 22.** Suppose that two segments are each equal to a third segment. How do these two segments compare with each other?

**Ex. 23.** Suppose that two segments are each equal to equal segments. How do these segments compare with each other?

**Ex. 24.** Complete the following sentences:
(*a*) If equal segments are added to equal segments, the sums are ...
(*b*) If equal segments are subtracted from equal segments, the remainders are ...




**14.** It will be assumed as apparent that: *the straight line-segment is the shortest line between two points.*

The **Distance** between two points is the length of the segment of the straight line between the points.

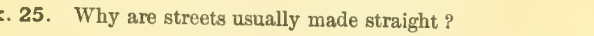
To obtain a straight line between two points, a carpenter stretches a piece of twine between the two points. In doing so, he assumes that the shortest line between two points is the straight line.

**Ex. 25.** Why are streets usually made straight?

**Ex. 26.** Why do people often "cut across" a vacant corner lot?

**Ex. 27.** Place upon paper points $A$, $B$, and $C$ so that they do not all lie upon a straight line. Draw segments $AB$, $BC$, and $AC$. By means of your dividers compare the longest segment with the sum of the other two segments.

**15. A point bisects a segment** if it divides the segment into two equal segments. The point is called the **Mid-point** of the segment.




Thus, $C$ bisects $AB$ if $AC = CB$.

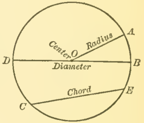
It will be assumed as apparent that *a segment has only one mid-point.*

**Ex. 28.** Determine with your dividers whether $C$ does actually bisect $AB$. If it does, what part of $AB$ is $AC$? Does $D$ bisect $AC$? Does $E$ bisect $CB$? (See Fig. § 15.)

**Ex. 29.** Draw a segment of any length and locate freehand the point which you think bisects the segment. Test the two parts to determine whether you actually located the mid-point of the segment. (Continue this exercise until you can approximately bisect a segment in this manner.)

**Ex. 30.** What must be true about halves of equal segments?

**16.** A **Circle** is a closed curved line all points of which are equidistant from a point within called the **Center.**




A **Radius** of a circle is the distance from the center to any point on the circle; as $OA$.

A **Diameter** of a circle is a segment drawn through the center of the circle with its ends on the circle; as $BD$.

A **Chord** of a circle is the segment joining any two points of the circle; as, chord $CE$.

A circle can be drawn with any point as center and any given segment as radius.

**17.** Two circles having equal radii can be made to coincide and are called **equal circles.** Hence:

*All radii of the same circle or of equal circles are equal.*

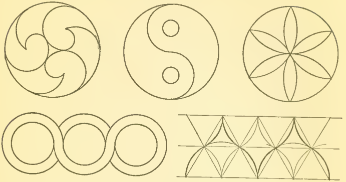
**Ex. 31.** Draw a circle of radius 1 inch.

**Ex. 32.** Draw two circles having the same center with radii of 1.5 in. and 2 in. respectively.

**Ex. 33.** Draw a circle and a straight line which intersects it. How many points of intersection are there?

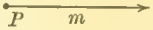
**Ex. 34.** Draw two circles that intersect. How many points of intersection are there?

**18.** Circles form the basis of numerous designs.



Can you copy any of these designs?

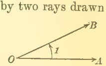
**19.** A **Half-line** or **Ray** is the part of a straight line in one direction from a given point on the line; as ray *m*.




**Ex. 35.** How many end-points has a line-segment? A ray? A line?

**20.** An **Angle** ($\angle$) is the figure formed by two rays drawn from the same point.

This definition was introduced by a mathematician, Bertrand, in 1778.




The common point is called the **Vertex** of the angle.

The two rays are called the **Sides** of the angle.

One may imagine a ray starting from the position $OA$ and turning about point $O$ until it occupies the position $OB$. $OA$ is then called the *initial* line, and $OB$ the *terminal* line.

NOTE 1. — The size of an angle may be thought of as depending upon the amount of turning about the vertex which a line must do to pass from the initial line to the terminal line.

NOTE 2. — The portion of the plane over which the line would pass is said to be within the angle. (See Note 2, page 28.)

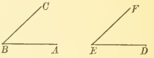
NOTE 3. — Since the rays extend indefinitely, it is clear that the size of an angle does not depend upon the length of its sides.

An angle may be named by the letter at its vertex if there is in the figure only one angle having that vertex; as $\angle O$.

An angle may be indicated by a number placed within the angle near its vertex; as $\angle 1$.

An angle may be named by reading the letters $A$, $O$, $B$; as $\angle AOB$, where the vertex letter $O$ is placed between the other two letters.

**21.** Two **angles** are **equal** if they can be made to coincide.



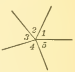

Thus, if $\angle E$ can be placed upon $\angle B$ so that point $E$ is on point $B$, line $ED$ on line $BA$, and line $EF$ on line $BC$, then $\angle E$ equals $\angle B$.

**22.** An $\angle AOB$ is less than $\angle AOC$ if it equals a part of $\angle AOC$.


**Ex. 36.** Make on thin paper a tracing of $\angle DEF$ (§ 21). Place the tracing over $\angle ABC$ and thus determine whether the angles are actually equal.

**Ex. 37.** What must be true about two angles each of which is equal to a third angle?

**Ex. 38.** In the adjoining figure, make a tracing of angle 1. Determine which of the other angles are equal to $\angle 1$.

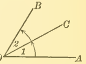



**23. A line bisects an angle** if it divides the angle into two equal angles.

Thus, $OC$ bisects $\angle AOB$ if $\angle 1 = \angle 2$.

The line is called the **Bisector** of the angle.

It will be assumed as apparent that *an angle has only one bisector.*




**Ex. 39.** Make a tracing of $\angle 1$ and determine whether $\angle 1$ is actually equal to $\angle 2$. Is $OC$ actually the bisector of $\angle AOB$?

**Ex. 40.** What part of $\angle AOB$ is $\angle 1$? (See Fig. § 23.)

**Ex. 41.** Draw any angle. Can you fold the paper so that the crease will bisect the angle?

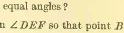
**Ex. 42.** Draw any angle. Draw a line which you think bisects the angle. Test the equality of the two parts of the angle by means of tracing paper. (Continue this exercise until you can approximately bisect au angle in this manner.)

**Ex. 43.** What must be true about halves of equal angles?

**Ex. 44.** Suppose that $\angle ABC$ is placed upon $\angle DEF$ so that point $B$ is on point $E$, and line $BA$ is on line $ED$.

Where will $BC$ fall if $\angle ABC$ is equal to $\angle DEF$?

Where will $BC$ fall if $\angle ABC$ is less than $\angle DEF$?




Where will $BC$ fall if $\angle ABC$ is greater than $\angle DEF$?

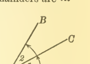
**Ex. 45.** Complete the following sentences:

(*a*) If equal $\angle$s are added to equal $\angle$s, the sums are ...

(*b*) If equal $\angle$s are subtracted from equal $\angle$s, the remainders are ...

**24. Adjacent Angles** are two angles that have a common vertex and a common side between them.

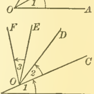
Thus, $\angle 1$ and $\angle 2$ are adjacent angles.




**Ex. 46.** In the adjoining figure:

(*a*) Is $\angle 1$ adjacent to $\angle 2$? Why?

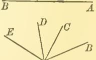
(*b*) Is $\angle 2$ adjacent to $\angle 3$? Why?




**25.** Two **adjacent angles** are readily **added.**

Thus, $\angle AOB + \angle BOC = \angle AOC$.

Also, $\angle AOC - \angle BOC = \angle AOB$.




**Ex. 47.** In the figure (§ 25), read tbe angle which represents:

(*a*) $\angle AOB + \angle BOD$;

(*b*) $\angle BOC + \angle COD$;

(*c*) $\angle BOC + \angle COD + \angle DOE$;

(*d*) $\angle BOE - \angle DOE$;

(*e*) $\angle AOC + \angle COD - \angle BOD$;

(*f*) $\angle AOE - \angle DOE - \angle COD$.

**26.** If one straight line meets another straight line so that the adjacent angles formed are equal, each of these angles is a **Right Angle**; as, $\angle 1$ and $\angle 2$.


**27.** It will be assumed as apparent that *all right angles are equal.*

**28.** An angle is measured by finding how many times it contains another angle selected as unit of measure.

The usual unit of measure is the **Degree**, which is one ninetieth of a right angle.

To express fractional parts of the unit, the degree is divided into sixty equal parts, called minutes, and the minute into sixty equal parts, called seconds.

Degrees, minutes, and seconds are represented by the symbols °, ′, and ″ respectively.

Thus, $42° 22′ 37″$ denotes an angle of 42 degrees, 22 minutes, and 37 seconds.

**Ex. 48.** Point out in your classroom some right angles.

**Ex. 49.** Fold a piece of paper so as to make a straight line; then fold it again so as to form two equal adjacent angles. Then open it out.

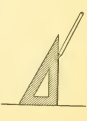
What kind of angles are formed by the lines along which the paper was folded?

**Ex. 50.** How many degrees are there in:
$\frac{1}{2}$ rt. $\angle$? $\frac{1}{3}$ rt. $\angle$? $\frac{1}{4}$ rt. $\angle$? $\frac{2}{3}$ rt. $\angle$?


**29.** Two lines are **Perpendicular** ($\perp$) if they form a right angle.

Thus, $BA \perp CD$ if $\angle 1 = \angle 2$.

When two lines are perpendicular, the adjacent angles are equal (§ 26).

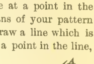


A practical method of drawing a perpendicular to a line at a point in the line, is to use a pattern right angle as in the adjoining figure. Draughtsmen use the right triangle illustrated. A card with two perpendicular edges may be used as well.

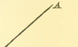
**Ex. 51**. Draw a perpendicular to a line at a point in the line as suggested in § 29. Use a card or else the pattern right angle constructed in Exercise 49.

**Ex. 52**. Draw a perpendicular to a line from a point not in the line, using a pattern right angle.

**Ex. 53**. Draw freehand a perpendicular to a line at a point in the line. Test the accuracy of your construction by means of your pattern right angle. (Continue this exercise until you can draw a line which is approximately perpendicular to a given line, either at a point in the line, or from a point not in the line.)



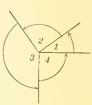

**30.** An **Acute Angle** is an angle which is less than a right angle; as $\angle CBA$.




An **Obtuse Angle** is an angle which is greater than a right angle; as $\angle FED$.

Acute and obtuse angles are called collectively **Oblique Angles.**

Two intersecting lines which are not perpendicular are said to be oblique to each other.

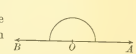



**Ex. 54**. What kind of angle is $\angle 1$? $\angle 2$? $\angle 3$? $\angle 4$? (Test each with your pattern right angle.)

**Ex. 55**. What kind of angle do the hands of a clock form at 3 o'clock? At 1 o'clock? At 2 o'clock? At 5 o'clock?

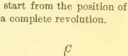
**Ex. 56**. How many degrees are there in each of the angles in Exercise 55?

**31.** A **Straight Angle** is an angle whose sides lie in a straight line on opposite sides of its vertex.




Such an angle would result if a line were to start from the position of line $OA$ and revolve about point $O$ one half of a complete revolution.

**32.** If $AB$ is any straight line, and $CO \perp AB$, then $\angle 1$ and $\angle 2$ are each right angles by § 29; also, $\angle AOB$ is a straight angle by § 31. Hence *a straight angle equals two right angles.*



**33.** Since a straight angle is equal to two right angles (§ 32) and since all right angles are equal (§ 27), it is evident that *all straight angles are equal.*

**Ex. 57.** How many degrees are there in a straight angle?

**Ex. 58.** What part of a straight angle is a right angle?

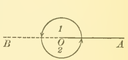
**Ex. 59.** At what hour do the hands of a clock form a straight angle?

**34.** *The sum of all the successive adjacent angles around a point on one side of a straight line is one straight angle.*

Thus, $\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle AOB = 1 \text{ st} \angle$.


**Ex. 60.** If $\angle 1 = \angle 2 = \angle 3 = \angle 4$, how many degrees are there in each angle?

**Ex. 61.** If $\angle 2 = 3$ times $\angle 1$, $\angle 3 = \angle 2$, and $\angle 4 = 2$ times $\angle 1$, how many degrees are there in each angle? (Use algebraic method.)




**35.** *The sum of all the successive adjacent angles around a point is two straight angles.*

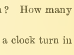
What kind of angle is $\angle 1$? $\angle 2$? Hence $\angle 1 + \angle 2 =$ ?

The total angular magnitude around a point is called a **Perigon.**

*Perigon* is from Greek words meaning "the angle around."

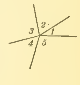
**Ex. 62.** How many right angles are there in a perigon? How many degrees?

**Ex. 63.** Through what angle does the minute hand of a clock turn in one half hour? In one hour?




**Ex. 64.** How large would each angle of the adjoining figure be if the angles were equal angles?

**Ex. 65.** How large would each angle of the adjoining figure be if $\angle 1$ were equal to $\angle 4$, if $\angle 3 = 2$ times $\angle 1$, if $\angle 5 =$ the sum of $\angle 3$ and $\angle 1$, and if $\angle 2 =$ the sum of $\angle 3$ and $\angle 5$?

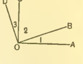



**36.** Two angles are **Complementary** if their sum is equal to a right angle. Each of the angles is called the **Complement** of the other.

Thus, the complement of 40° is 50°.

**Ex. 66.** What is the complement of 10°? of 25°? of 50°? of 90°? of 0°? of 45°? of $x°$?

**Ex. 67.** How large is the angle which equals its complement? (Use algebraic method.)

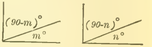



**Ex. 68.** How large is the angle which equals four times its complement?

**Ex. 69.** If $OC \perp OA$ and $OD \perp OB$, and if $\angle 2 = 70°$, compare $\angle 1$ and $\angle 3$.

**Ex. 70.** Draw any acute angle. Through the vertex draw a line which will form with one side of the angle the complement of the angle.

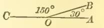
**37.** *Complements of the same angle or of equal angles are equal.*




The complement of $m°$ is $(90 - m)°$ and the complement of $n°$ is $(90 - n)°$. If, now, $m = n$, then $90 - m$ must equal $90 - n$, for when equals are subtracted from equals the remainders are equal.

**38.** Two angles are **Supplementary** if their sum is equal to a straight angle. Each of the angles is called the **Supplement** of the other.

Thus, an angle of 150° is the supplement of an angle of 30°.



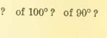

**Ex. 71.** What is the supplement of 80°? of 60°? of 100°? of 90°? of 0°? of $x°$? of $3x°$?

**Ex. 72.** How large is the angle which equals its supplement?

**Ex. 73.** How large is the angle which is nine times as large as its supplement?

**Ex. 74.** Draw any angle less than a straight angle. Through its vertex draw a straight line which will form with one side of the angle the supplement of the angle.

**39.** *If two adjacent angles have their exterior sides in a straight line, they are supplementary.*




In the adjoining figure, $\angle 1$ and $\angle 2$ are adjacent angles. $OC$ and $OA$ are their exterior sides. If $OC$ and $OA$ lie in a straight line, then $\angle AOC = 1$ st. $\angle$. But $\angle 1 + \angle 2 = \angle AOC$. Hence, $\angle 1$ and $\angle 2$ are supplementary (§ 38).

Two adjacent angles which are also supplementary are called **Supplementary-adjacent Angles**.

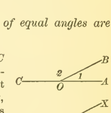
**40.** *If two adjacent angles are supplementary, their exterior sides lie in a straight line.* (Fig. of § 39.)

For, if $\angle 1 + \angle 2 = 1$ st. $\angle$, then $\angle AOC$ must be a straight angle and $AC$ must be a straight line.

**Ex. 75**. Draw two adjacent angles which are not supplementary.

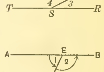
**41.** *Supplements of the same angle or of equal angles are equal.*

In the adjoining figure, let $\angle 1 = \angle 3$, and let $AC$ and $RT$ be straight lines. Then $\angle 2$ is the supplement of $\angle 1$ and $\angle 4$ is the supplement of $\angle 3$; that is, $\angle 2 = 180 - \angle 1$ and $\angle 4 = 180 - \angle 3$. Clearly, then, $\angle 2$ must equal $\angle 4$, for, when equal angles are subtracted from equal angles, the remainders are equal.

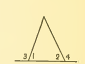



**Ex. 76**. If $AB$ and $CD$ are straight lines, are $\angle 1$ and $\angle 2$ supplementary? $\angle 3$ and $\angle 4$? Why? (See § 39.)

Suppose that $\angle 1 = \angle 4$. Must $\angle 3$ then equal $\angle 2$? Why?




**Ex. 77**. In the adjoining figure, if $\angle 1$ equals $\angle 2$, then $\angle 3$ must equal $\angle 4$. Prove it.




**42.** Two angles are called **Vertical Angles** when the sides of one are prolongations of the sides of the other.

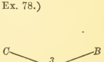
Thus, $\angle 1$ and $\angle 2$ are vertical angles. (See Fig. Ex. 78.)

**Ex. 78**. If $\angle 1 = 40°$, how many degrees are there in $\angle 3$?

How many degrees are there in $\angle 2 + \angle 3$?

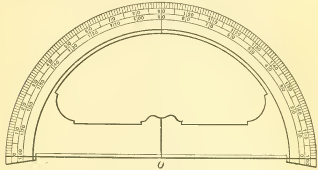
How many degrees are there in $\angle 2$?

How then do $\angle 2$ and $\angle 1$ compare?



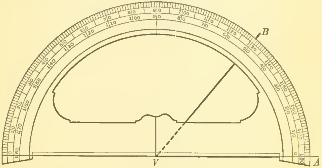
**Ex. 79.** Draw two straight lines that intersect. Make a tracing of one of the angles formed and compare it with its vertical angle. What do you conclude must be true about the vertical angles?

**43.** The **Protractor** is a tool for measuring angles.




The point $O$ on the protractor will be referred to as the center of the protractor.

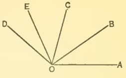
**44. Problem.** *Measure a given angle AVB.*




Place the protractor over the given angle so that its center is on the vertex, $V$, of the angle, and its edge is on $VA$. Then read from the protractor the number opposite the point where $VB$ crosses the outer edge of the protractor. This gives the number of degrees in the angle $AVB$. Thus, in the figure, $\angle AVB = 50°$.

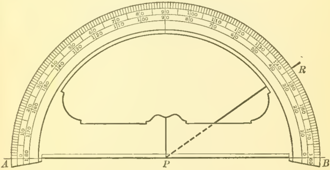
**Ex. 80.** Make on paper a tracing of the adjoining figure. On your paper, extend rays $OA$, $OB$, $OC$, $OD$, and $OE$ until they are about 3 in. in length; then measure:

(*a*) $\angle AOB$; (*b*) $\angle AOC$; (*c*) $\angle BOE$; (*d*) $\angle COD$.




**45. Problem.** *Construct an angle of given size at a point in a given line.*

Construct an angle of 35° at $P$ in $AB$.




Place the protractor with its center on $P$ and its edge on $PB$ as in the figure. Then place a point $R$ on the paper opposite the 35° mark on the protractor. Remove the protractor and draw the line $PR$. Then angle $BPR$ equals 35°.

**Ex. 81.** Construct with the protractor an angle of:

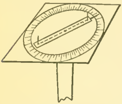
(*a*) 70°; (*b*) 40°; (*c*) 65°; (*d*) 100°; (*e*) 143°.

Write below each angle whether it is an acute or an obtuse angle.

**46. A Field Protractor.** The figure at top of page 20 represents a simple field protractor which can be made by some member of the class. With it angles can be measured in the field and thus some elementary surveying problems can be solved.

On a flat board about 20 inches square, draw a circle of diameter 10 inches. Divide its circumference into 360 equal parts.

Make an arm which may swing about the center of the circle as pivot. Let the arm have upon it two "sights" directly in line with the center of the circle. At the end of the arm and in line with the sights place a pin.



The board may be attached to the end of a stake about 4 feet long, or better to a tripod.

This instrument can be used to measure angles in the open field.

Thus, to measure an $\angle CAB$, place the instrument over point $A$. Make the board stand level. (An inexpensive level would be a great help.) Holding the board stationary, sight first at point $C$, and read the angle on the protractor; then sight at point $B$, and note the angle on the protractor. The number of degrees through which the arm is turned in passing from $AC$ to $AB$ is the measure of angle $CAB$.

**47. Triangle** ($\triangle$). Three points which do not lie in the same straight line determine three segments. Thus, $A$, $B$, and $C$ determine the segments $AB$, $BC$, and $AC$. The figure formed by these three segments is called the triangle $ABC$ ($\triangle ABC$). $A$, $B$, and $C$ are the *vertices* of the triangle; $AB$, $BC$, and $AC$ are the *sides* of the triangle; $\angle A$, $\angle B$, and $\angle C$ are the *angles* of the triangle.


The sides and angles of the triangle are called the *parts* of the triangle. They are six in number. Opposite each side there is an angle, and opposite each angle there is a side. Thus, $\angle C$ is opposite side $AB$.

**48. Experimental Geometry.** Many facts about geometrical figures can be discovered by careful drawing, measurement, and observation.

**Ex. 82.** On a line $AB$, at a point $P$, draw a ray $PC$ making $\angle APC = 80°$. Measure $\angle CPB$. What fact studied previously does this exercise verify?

**Ex. 83.** Draw two intersecting straight lines. Measure each of a pair of vertical angles. What fact studied previously does this exercise verify?

**Ex. 84.** Construct a $\triangle ABC$, having $AB$ and $BC$ each 3 inches long and $\angle B = 40°$. Measure $\angle A$ and $\angle C$. How do they compare? (A triangle having two equal sides is called an isosceles triangle.)

**Ex. 85.** Draw any triangle having two equal sides. Measure the angles opposite the equal sides. How do they compare? What fact is suggested by Exercises 84 and 85?

**Ex. 86.** Draw any triangle of reasonably large size. Measure each of its angles. Find their sum. Repeat the exercise for another triangle of somewhat different shape. Compare your results with those of some other pupils. What seems to be the sum of the angles of a triangle?

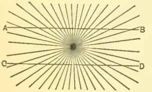
**Ex. 87.** Draw a $\triangle ABC$ in which $AB$ and $BC$ are each 3 inches and $\angle B = 50°$. Let $E$ be the mid-point of $BC$, and $F$ the mid-point of $AB$. Draw $AE$ and $CF$. Measure them. What seems to be true?

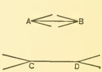
**Ex. 88.** Draw any triangle $ABC$ of reasonably large size. Let $E$ be the mid-point of $AB$, and $F$ the mid-point of $BC$. Draw $EF$. Compare $EF$ and $AC$ by measuring them. What seems to be the relation between them?

**Ex. 89.** Draw a segment $AB$. At its center, $C$, draw a line $CD$ perpendicular to $AB$. From $E$, any point on $CD$, draw $AE$ and $BE$. Compare them by means of your dividers. Take any other point on $CD$ and repeat the exercise. What fact seems to be suggested?

**Ex. 90.** Let $AB$ be any line segment. Draw $CA \perp AB$ at $A$, and $DB \perp AB$ at $B$. Make $CA = DB$. Draw $AD$ and $CB$. Compare them either by measurement or by means of the dividers.

**49.** Objections to studying geometry *only by the experimental method* may be given. To get satisfactory results, the figures must be drawn and measured with greater accuracy than is usually possible. Conclusions reached from the study of one or two special figures may be incorrect. *Frequently one is misled by assuming relations which appear to the eye to be correct.*




**Ex. 91.** In the first figure above, are $AB$ and $CD$ straight lines?

**Ex. 92.** In the second figure above, is $AB$ equal to or less than $CD$?

**50. Demonstrative Geometry.** For the reasons given in § 49 and for other reasons, it is customary to study geometry by what is known as the demonstrative method. *Statements are not accepted until they are proved to be true*, except for a few fundamental ones which are assumed as a foundation.

**51.** An **Axiom** is a statement accepted as true without proof. Usually the truth is very evident. The following are important axioms; others will be introduced as they become necessary.

AXIOMS

Ax. 1. *Quantities which are equal to the same quantity or to equal quantities are equal to each other.* (See Ex. 22 and Ex. 23.)

Ax. 2. *Any quantity may be substituted for its equal in a mathematical expression.*

Ax. 3. *If equals be added to equals, the sums are equal.* (See Ex. 24.)

Ax. 4. *If equals be subtracted from equals, the remainders are equal.*

Ax. 5. *If equals be multiplied by equals, the products are equal.*

Ax. 6. *If equals be divided by equals, the quotients are equal.* (The divisor must not be zero.)

Ax. 7. *The whole equals the sum of its parts.*

Ax. 8. *The whole is greater than any of its parts.*

Ax. 9. *If $a$ and $b$ are any two magnitudes of the same kind, then $a$ is less than $b$, is equal to $b$, or is greater than $b$.*

Ax. 10. *Only one straight line can be drawn through two points.* (§ 8.)

Ax. 11. *The straight line segment is the shortest line that can be drawn between two points.* (§ 14.)

Ax. 12. *All right angles are equal.* (§ 27.)

Ax. 13. *An angle has only one bisector.* (§ 23.)

Ax. 14. *A segment has only one mid-point.* (§ 15.)

**52.** A **Theorem** is a statement which requires proof. Every theorem can be expressed by a sentence which has one clause beginning with "if" and a second clause beginning with "then."

The clause beginning with "if" is called the **Hypothesis**; it indicates what is known or assumed.

The clause beginning with "then" is called the **Conclusion**; it states what is to be proved.

Thus: (Hypothesis) *If two sides of a triangle are equal,*
(Conclusion) *then the angles opposite are equal.*

**53.** Some theorems have been proved already in an informal manner.

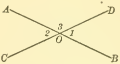
INFORMALLY PROVED THEOREMS

1. *Two straight lines can intersect at only one point.* (§ 11.)
2. *All radii of the same circle or of equal circles are equal.* (§ 17.)
3. *A straight angle equals two right angles.* (§ 32.)
4. *All straight angles are equal.* (§ 33.)
5. *The sum of all the successive adjacent angles around a point on one side of a straight line is one straight angle.* (§ 34.)
6. *The sum of all the successive adjacent angles around a point is two straight angles.* (§ 35.)
7. *Complements of the same angle or of equal angles are equal.* (§ 37.)
8. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (§ 39.)
9. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.* (§ 40.)
10. *Supplements of the same angle or of equal angles are equal.* (§ 41.)

**54.** In a formal demonstration or proof, each statement made is proved by quoting a definition, an axiom, the hypothesis, or some previously proved theorem.

### ILLUSTRATIVE DEMONSTRATION

THEOREM. *If two straight lines intersect, the vertical angles are equal.*




**Hypothesis.** St. lines $AB$ and $CD$ intersect at $O$, forming vertical $\angle$s 1 and 2.

**Conclusion.** $\angle 1 = \angle 2$.

| | | |
|---|---|---|
| **Proof.** 1. | $AB$ is a straight line. | Hyp. |
| 2. | $\therefore \angle 1$ is a supplement of $\angle 3$. | § 39 |
| | [If two adj. $\angle$s have their ext. sides in a st. line, they are supplementary.] | |
| 3. | $CD$ is a straight line. | Hyp. |
| 4. | $\therefore \angle 2$ is a supplement of $\angle 3$. | § 39 |
| 5. | $\therefore \angle 1 = \angle 2$. | § 41 |
| | [Supplements of the same $\angle$ are equal.] | |

NOTE. — This theorem was apparently assumed by Thales.

**Ex. 93.** Prove in the same manner that $\angle AOD = \angle COB$.

**Ex. 94.** If $\angle 3 = 130°$, how many degrees are there in each of the other angles in the figure above?

**Ex. 95.** Two lines intersect so that one of the angles is a right angle. How large is each of the other angles formed?

**55.** Besides the proofs of certain theorems, the methods of constructing certain figures are studied in geometry.

A **Problem** is a construction to be made.

**56.** A **Postulate** is a construction assumed possible.

The following postulates are necessary at the present time.

### POSTULATES

1. *One straight line can be drawn through two points.* (§ 8.)
2. *A straight line can be extended indefinitely in each direction.* (§ 8.)
3. *A circle can be drawn with any point as center and any given segment as radius.*

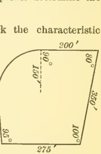
**57.** The word **Proposition** is commonly used to designate a theorem or a problem discussed in the text.

### SUPPLEMENTARY EXERCISES

**Ex. 96.** Draw a line to represent the path of a baseball when the pitcher throws an "out-curve."

**Ex. 97.** A farmer is setting out trees for an orchard. He first places the trees which are at the ends of a row. How may he then locate the other trees of that row so that they will be in a straight line without using a long line between the two end trees? What two points determine the straight line formed by the trees?

**Ex. 98.** How do plasterers use in their work the characteristic property of a plane mentioned in § 1?




**Ex. 99.** A man wishes a scale drawing of his suburban lot so that he may consult a landscape architect about the proper planting of it. He made the adjoining rough drawing of the lot, and then obtained the measurements indicated. Make a scale drawing of the lot, letting $\frac{1}{4}$ inch represent 25 feet.

#### Supplementary and Complementary Angles

**Ex. 100.** What is the complement of 40° 30′?

**Ex. 101.** What is the supplement of 56° 30′?

**Ex. 102.** How many degrees are there in an angle if its complement contains 40°?

**Ex. 103.** How many degrees are there in an angle whose supplement contains 80°?

**Ex. 104.** Find the angle whose supplement is ten times its complement.

**Ex. 105.** Two angles are complementary. The greater exceeds the less by 25°. Find the angles. (Use the algebraic method.)

**Ex. 106.** Find the angle which exceeds its supplement by 34°.

**Ex. 107.** The sum of the supplement and complement of a certain angle is 140°. Find the angle.

**Ex. 108.** Find the number of degrees in the angle the sum of whose supplement and complement is 196°.

**Ex. 109.** The supplement of a certain angle exceeds three times its complement by 18°. Find the angle.

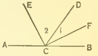
**Ex. 110.** Prove that the supplement of any angle exceeds its complement by one right angle.

### Vertical Angles

**Ex. 111.** Prove that the straight line which bisects an angle also bisects the vertical angle.


**Hyp.** $OE$ bisects $\angle AOC$. $EOF$ is a st. line.
**Con.** $OF$ bisects $\angle BOD$.

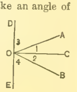
**Ex. 112.** Prove that the bisectors of two supplementary adjacent angles are perpendicular.




*Suggestions.* — $\angle 1 = \frac{1}{2}\angle BCD$; $\angle 2 = \frac{1}{2}\angle DCA$.
$\therefore \angle 1 + \angle 2 = ?$

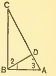
**Ex. 113.** If the bisectors of two adjacent angles are perpendicular, the angles are supplementary. (See figure of Ex. 112.)

**Ex. 114.** If the bisectors of two adjacent angles make an angle of 45°, the angles are complementary.




**Ex. 115.** **Hyp.** $CO$ bisects $\angle AOB$. $DE \perp CO$.
**Con.** $\angle 3 = \angle 4$.

*Suggestions.* — 1. Compare $\angle 1$ and $\angle 2$.
2. Compare $\angle 3$ and $\angle 1$; $\angle 4$ and $\angle 2$.




**Ex. 116.** **Hyp.** $\angle ABC$ is a rt. $\angle$.
$\angle 3$ is complementary to $\angle 1$.
**Con.** $\angle 3 = \angle 2$.

**Ex. 117.** Two straight lines intersect so that one angle formed is 60°. How large is each of the other angles?

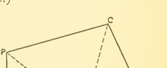
**Ex. 118.** Review the following definitions:

(*a*) Segment of a straight line.
(*b*) Equal segments.
(*c*) Mid-point of a segment.
(*d*) Circle.
(*e*) Radius and diameter.
(*f*) Ray or half-line.
(*g*) Angle; vertex; side.
(*h*) Equal angles.
(*i*) Bisector of an angle.
(*j*) Right angle; acute; obtuse; straight.
(*k*) Complementary angles; supplementary; adjacent; vertical.
(*l*) Perpendicular lines.

**Ex. 119.** What is an axiom? a theorem?

**Ex. 120.** What is the hypothesis of a theorem? What is the conclusion of a theorem?

**Ex. 121.** Lay off on a field some irregular piece of ground. Obtain such measurements as will enable you to make a scale drawing of the field. (This exercise is similar to Ex. 99.)




**Ex. 122.** Another method for obtaining measurements for a scale drawing for a piece of ground and objects upon it is to locate the instrument for measuring angles at a point like $O$ in the adjoining figure. Then find the distances of the other points from $O$ and their directions from $OD$ or from $OA$.

Select an irregular piece of ground and obtain the measurements which will enable you to make a scale drawing of it and locate upon the drawing some of the trees or other objects on the lot.

#### Supplementary Notes on Definitions

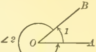
NOTE 1. — *Point and straight line are undefined.* (See § 4 and § 5.) A definition describes a term by means of simpler terms. It is evident

then that there must be some terms which cannot be defined, as there are no terms simpler than them by which to define them.

No definition of point can be given.

No satisfactory definition of straight line suitable for high school pupils can be given.

Hence point and straight line are left undefined.



NOTE 2. — *Relating to the definition of an angle* (§ 20).

Two rays $OA$ and $OB$ *actually* form *two* angles; namely, $\angle 1$ and $\angle 2$ of the adjoining figure. In $\angle 1$, $OA$ is the initial line (§ 20) and $OB$ is the terminal line; in $\angle 2$, $OB$ is the initial line and $OA$ is the terminal line. Usually, one angle is less than and the other is greater than a straight angle.

Unless something is said to tbe contrary, $\angle AOB$ refers to the smaller angle formed by the rays $OA$ and $OB$.

# BOOK I

## RECTILINEAR FIGURES

**58.** A **Rectilinear Figure** is a geometrical figure composed of straight lines only.

**59. Congruence.** If asked to compare two sheets of paper as to shape and size, it is natural to place one upon the other to determine whether they can be made to coincide (fit together).

**60.** Two **geometrical figures** are **congruent** ($\cong$) if they can be made to coincide.

**61.** Ax. 15. **Congruence Axiom.** — *Two figures which are congruent to the same figure are congruent to each other.*

**Historical Note.** — The symbol $\cong$ was introduced by a mathematician, Leibnitz, in 1679.

**62. Superposition** is the process of placing one geometrical figure upon another for the purpose of comparing them. Literally, superpose means "place above."

**Postulate.** — *A geometrical figure may be moved about in space without changing any of its parts.*

**Ex. 1.** Notice the panes of glass in the windows of your schoolroom. Do they appear to be congruent? Do they coincide now?

**Ex. 2.** Draw $\triangle ABC$, having $AB = 4$ in., $BC = 6$ in., and $\angle B = 50°$.

(*a*) Measure $\angle A$, $\angle C$, and $AC$.

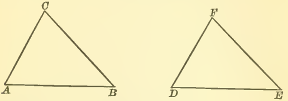
(*b*) Cut your triangle from the paper. Compare it by superposition with the triangles made by other members of your class.

(*c*) What do you conclude must be true about all triangles made according to the directions?

**Ex. 3.** Are the statements of the hypothesis assumed to be true or must they be proved to be true? Answer the same question for the conclusion.

## Proposition I. Theorem

**63.** *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.*



**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $AC = DF$; $\angle A = \angle D$.

**Conclusion.** $\triangle ABC \cong \triangle DEF$.

**Proof.** 1. Place $\triangle ABC$ on $\triangle DEF$ with point $A$ on point $D$, and side $AB$ on side $DE$.

2. Point $B$ will fall on point $E$.
[Since $AB = DE$, by hypothesis.] § 13

3. $AC$ will fall on $DF$.
[Since $\angle A = \angle D$, by hypothesis.] § 21

4. Point $C$ will fall on point $F$.
[Since $AC = DF$, by hypothesis.] § 13

5. $\therefore BC$ must coincide with $EF$.
[Only one st. line can be drawn through two points.] Ax. 10; § 51

6. $\therefore \triangle ABC$ coincides with $\triangle DEF$ and is congruent to it.
[Two $\triangle$s are congruent if they can be made to coincide.] § 60

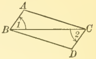
**Ex. 4.** After you place $\triangle ABC$ so that $A$ falls on $D$, does $AB$ fall on $DE$, or must you place $AB$ on $DE$?

**Ex. 5.** Where would $C$ fall if $AC$ were equal to $\frac{1}{2}DF$? Would the triangles be congruent?

**Ex. 6.** Where would $AC$ fall if $\angle A$ were less than $\angle D$?

**Ex. 7.** Make a free-hand drawing to illustrate the result if a $\triangle ABC$ is superposed on a $\triangle DEF$, when $AC = \frac{1}{2}DF$, $\angle A = \angle D$, and $AB = \frac{2}{3}DE$.

**64. Application of First Theorem. Illustrative Exercise.**




**Hypothesis.** $AB = CD$.
$\angle 1 = \angle 2$.

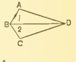
**Conclusion.** $\triangle ABC \cong \triangle BCD$.

**Proof.** 1. In $\triangle ABC$ and $\triangle BCD$:

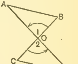
| | | |
|---|---|---|
| | $AB = CD$; | Hyp. |
| | $\angle 1 = \angle 2$; | Hyp. |
| | $BC \equiv BC$. | See note below. |
| 2. | $\therefore \triangle ABC \cong \triangle BCD$. | |

[If two $\triangle$s have two sides and the included $\angle$ of one eqnal respectively to two sides and the included $\angle$ of the other, the $\triangle$s are congruent.]

**Note.** — The symbol $\equiv$ is read "is identically equal to." No other authority is required, for any magnitude is equal to itself.



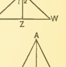

**Ex. 8.** Hyp. $\angle 1 = \angle 2$; $AB = BC$.
Con. $\triangle ABD \cong \triangle BCD$.




**Ex. 9.** Hyp. $AD$ and $BC$ are st. lines.
$AO = OD$; $BO = OC$.
Con. $\triangle ABO \cong \triangle CDO$.

*Suggestion.* — What do you know about $\angle 1$ and $\angle 2$? Why?


**Ex. 10.** Hyp. $XY = XW$.
$XZ$ bisects $\angle X$.
Con. $\triangle XYZ \cong \triangle XZW$.




**Ex. 11.** Hyp. $DBC$ is a st. line.
$AB \perp DC$; $DB = BC$.
Con. $\triangle ADB \cong \triangle ABC$.

**65. Homologous parts of congruent figures** are parts which are similarly located; they are the parts which coincide when

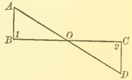
the figures are made to coincide. It follows that: ***homologous parts of congruent figures are equal.***

Thus, in § 63, $\angle C$ is homologous to $\angle F$, and side $BC$ is homologous to side $EF$. It follows that $\angle C = \angle F$ and $BC = EF$.

**Note.** — In congruent triangles, homologous sides lie opposite equal angles and homologous angles lie opposite equal sides.

**66. Principle I.** To prove that two segments are equal or two angles are equal, try to prove them homologous parts of congruent triangles.

Illustrative Exercise




**Hyp.** $BC$ is a st. line.
$AB \perp BC$; $DC \perp BC$;
$AB = DC$.
$O$ is mid-point of $BC$.

**Con.** $AO = OD$.

**Plan.** Try to prove $AO$ and $OD$ homologous sides of congruent $\triangle$s.

| | | |
|---|---|---|
| **Proof.** 1. | Since $AB \perp BC$, $\angle 1 =$ a rt. $\angle$. | Def. |
| 2. | Since $DC \perp BC$, $\angle 2 =$ a rt. $\angle$. | Def. |
| 3. | $\therefore \angle 1 = \angle 2$. [All rt. $\angle$s are equal.] | |
| 4. | In $\triangle ABO$ and $\triangle DCO$: | |
| | $AB = DC$; | Hyp. |
| | $\angle 1 = \angle 2$; | Step 3 |
| | $BO = OC$. | Hyp. |
| 5. | $\therefore \triangle ABO \cong \triangle DCO$. | |

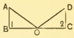
[If two $\triangle$s have two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other, the $\triangle$s are congruent.]

6. $\therefore AO = OD$.

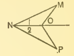
[Homologous sides of cong. $\triangle$s are equal.]

**Note.** — $AO$ lies opposite $\angle 1$ and $OD$ lies opposite $\angle 2$; hence they are homologous sides of the congruent triangles.

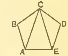
**Ex. 12.** Hyp. $AB \perp BC$; $DC \perp BC$;
$O$ bisects $BC$; $AB = DC$.
Con. $AO = OD$.



**Ex. 13.** Hyp. $NO$ bisects $\angle PNM$.
$MN = NP$.
Con. $\angle M = \angle P$.



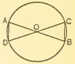
**Ex. 14.** Hyp. $AB = BC = CD = DE$.
$\angle B = \angle D$.
Con. $AC = CE$.



**Ex. 15.** To obtain the distance $AB$.

(1) Locate point $O$ from which $OA$ and $OB$ may be measured. (2) Extend $AO$ and $BO$, making $OC = AO$ and $OD = BO$. Then $DC = AB$. Prove it.

**Ex. 16.** If $AB$ and $CD$ are two diameters of a circle, prove that $AD$ must equal $BC$.



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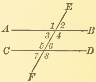
## Review Exercises



**Ex. 17.** If, in the adjoining figure, $\angle 3 = \angle 7$, prove $\angle 2 = \angle 7$.

**Ex. 18.** If $\angle 4 = \angle 5$, prove $\angle 1 = \angle 8$.

**Ex. 19.** If $\angle 3 = \angle 6$, prove $\angle 1 = \angle 5$.

*Suggestions.* —1. Recall § 41. 2. Of what angle is $\angle 1$ the supplement? 3. Of what angle is $\angle 5$ the supplement?

**Ex. 20.** If $\angle 4 = \angle 8$, prove $\angle 3 = \angle 6$.

**Ex. 21.** When are two figures congruent?

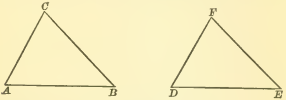
**Ex. 22.** What method of proof is employed in proving Proposition I?

---

**Ex. 23.** Draw a $\triangle ABC$, having $AB = 4$ in., $\angle A = 60°$, and $\angle B = 80°$. Cut your triangle from the paper and compare it with the triangles made by other members of your class. What do you conclude must be true about all triangles made according to the directions given?

## Proposition II. Theorem

**67.** *If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.*




**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $\angle A = \angle D$; $\angle B = \angle E$.

**Conclusion.** $\triangle ABC \cong \triangle DEF$.

**Proof.** 1. Place $\triangle ABC$ on $\triangle DEF$, with point $A$ on point $D$ and $AB$ on $DE$.

2. Point $B$ will fall on point $E$.
[Since $AB = DE$, by hypothesis.] § 13

3. $AC$ will fall on $DF$, $C$ falling somewhere on line $DF$.
[Since $\angle A = \angle D$, by hypothesis.] § 21

4. $BC$ will fall on $EF$, $C$ falling somewhere on line $EF$.
[Since $\angle B = \angle E$, by hypothesis.] § 21

5. $\therefore$ point $C$ must fall on point $F$.
[Two st. lines can intersect at only one point.] § 11

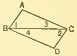
6. $\therefore$ $\triangle ABC$ coincides with $\triangle DEF$ and is congruent to it.
[Two ≙ are congruent if they can be made to coincide.] § 60

**Ex. 24.** Where would $AC$ fall if $\angle A$, above, were less than $\angle D$?

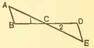
**Ex. 25.** Draw freehand the approximate figure which would result from superposing $\triangle ABC$ on $\triangle DEF$ if $AB = DE$, $\angle A = \frac{1}{2}\angle D$, and $\angle B = \frac{1}{2}\angle E$.

**Ex. 26.** After proving $\triangle ABC$ congruent to $\triangle DEF$, what do you know about: (*a*) $\angle C$? Why? (*b*) About $AC$? (*c*) About $BC$? § 65

**Ex. 27.** In Propositions I and II, how many parts (§ 47) of one triangle are given equal to parts of the other triangle?

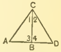



**Ex. 28.** Hyp. $\angle 1 = \angle 2$.
$\angle 3 = \angle 4$.
Con. $\triangle ABC \cong \triangle BCD$.



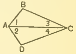

**Ex. 29.** Hyp. $AE$ and $BD$ are st. lines.
$\angle B = \angle D$; $C$ bisects $BD$.
Con. $\angle A = \angle E$.

*Suggestion.* — Read § 66.




**Ex. 30.** Hyp. $BC$ bisects $\angle C$.
$BC \perp AD$.
Con. $AB = BD$.

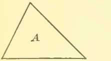
**Ex. 31.** If the line which joins two opposite vertices of a quadrilateral (four-sided figure) bisects the angles whose vertices it joins, then the other two angles are equal.




Hyp. $\angle 1 = \angle 2$; $\angle 3 = \angle 4$.
Con. $\angle B = \angle D$.

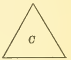
**Note.** — Supplementary Exercises 1 and 2, p. 273, can be studied now.

**68.** A triangle is **Scalene** when no two of its sides are equal; **Isosceles** when two of its sides are equal; **Equilateral** when all its sides are equal; **Equiangular** when all its angles are equal.




SCALENE


ISOSCELES



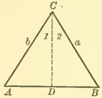

EQUILATERAL

A triangle can be made to "stand upon" any one of its sides. Hence any side of a triangle can be considered its **Base**. When a side has been selected as base, the opposite vertex is called the **Vertex** of the triangle, and the angle at that vertex is called the **Vertical Angle** of the triangle.

In an isosceles triangle, the side which is not one of the equal sides is usually taken as the base; and then the angle formed by the equal sides is the vertical angle of the isosceles triangle.

## Proposition III. Theorem

**69.** *In an isosceles triangle, the angles opposite the equal sides are equal.*



**Hypothesis.** In $\triangle ABC$, $AC = BC$.

**Conclusion.** $\angle A = \angle B$.

**Plan.** Prove $\angle A$ and $\angle B$ homologous $\angle$s of cong. $\triangle$s.

**Proof.** 1. Let $CD$ bisect $\angle C$.

2. In $\triangle ACD$ and $\triangle BCD$:

$AC = BC$; Hyp.

$CD \equiv CD$. See Note 1, § 64.

$\angle 1 = \angle 2$. Def.

[Since $CD$ bisects $\angle C$.]

3. $\therefore \triangle ACD \cong \triangle BCD$.

[If two $\triangle$s have two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other, the $\triangle$s are congruent.] § 63

4. $\therefore \angle A = \angle B$.

[Homologous $\angle$s of cong. $\triangle$s are equal.] § 65

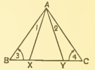
**Note 1.** — Principle I (§ 66) is used. To get two triangles, a *construction line*, *CD*, was drawn. This is often necessary.

**Note 2.** — This theorem is ascribed to Thales, although this may not be his proof. The proof for this theorem which was given by Euclid appears as Ex. 3, p. 273. It can be studied after § 73.

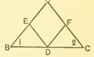
**Ex. 32.** Why are $\angle A$ and $\angle B$ homologous angles of the congruent triangles in the proof of Proposition III?

**70. Cor.** *An equilateral triangle is also equiangular.*
(Read § 71 at this time.)

**71.** A **Corollary** is a theorem which is easily deduced from the theorem with which it is given. For each corollary, draw a figure, form the hypothesis and conclusion, and give the proof.




**Ex. 33.** **Hyp.** $AB = AC.$
$\angle 1 = \angle 2.$
**Con.** $\triangle ABX \cong \triangle AYC.$
*Suggestion.* — Does $\angle 3 = \angle 4$? Why?

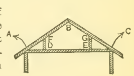



**Ex. 34.** If $AB = AC$ and if $D$ is the mid-point of $BC$, $E$ of $AB$, and $F$ of $AC$, prove that $ED = FD$.

*Suggestions.* — Form the hypothesis and conclusion. Read § 66. Does $BE = CF$? Why?

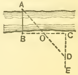
**Ex. 35.** After proving $DE = DF$ in Ex. 34, draw $EF$ and prove that $\angle DEF = \angle DFE$.

**Ex. 36.** In the figure drawn for Ex. 35, prove that $\angle AFE = \angle AEF$.



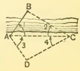

**Ex. 37.** If $AB$ and $CB$ are two rafters of equal length in a roof, and if $DF$ and $EG$ are supports, perpendicular to the floor $AC$, at points equally distant from $A$ and $C$ respectively, prove that $DF$ must equal $EG$. (Form the hypothesis and conclusion first.)

**Ex. 38.** In an isosceles triangle $ABC$, having $AB = AC$, point $D$ is any point in the base $BC$. $E$ is taken on $AC$ and $F$ on $AB$ so that $EC = BD$ and $BF = DC$. Prove that $DE = DF$. (Draw the figure as it is described.)




**Ex. 39.** To obtain the distance $AB$.

(1) Lay off $BC \perp$ to $AB$. (2) Lay off $CE \perp$ to $BC$. (3) Place a stake at $O$, the mid-point of $BC$. (4) Determine, by sighting, a point $D$ on $CE$ so that $A$, $O$, and $D$ will be in the same straight line. Then $CD = AB$. Prove it.

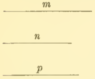


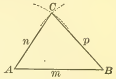

**Ex. 40.** To obtain the distance $AB$.

(1) Let $AC$ be any convenient segment. (2) Lay off $AD$, making $\angle 3 = \angle 1$. (3) Lay off $CD$, making $\angle 4 = \angle 2$. Then $AB = AD$. Prove it.

## Proposition IV. Problem

**72.** *Construct a triangle, having given its three sides.*




**Given** $m$, $n$, and $p$, the three sides of a triangle.

**Required** to construct the triangle.

**Construction.** 1. Draw $AB = m$.

2. With $A$ as center, and $n$ as radius, draw an arc.

3. With $B$ as center, and $p$ as radius, draw a second arc, intersecting the first arc at $C$.

4. Draw $AC$ and $BC$.

**Statement.** $\triangle ABC$ is the required triangle, as it has the given sides.

**Discussion.** If one side is equal to or greater than the sum of the other two sides, the construction is impossible.

**Ex. 41.** A piece of ground is triangular in form. Its sides measure 100 rd., 150 rd., and 200 rd., respectively. Make a scale drawing of the triangle, letting 1 in. represent 100 rd.

**Ex. 42.** Construct an isosceles triangle whose base is 2 in. and whose equal sides are each 3 in.

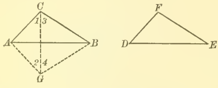
**Ex. 43.** A girl wants an equilateral triangle whose sides are each 3 in. long, to be used as a pattern in making a patch-work pillow-cover. Construct the equilateral triangle.

**Ex. 44.** Try to construct a triangle whose sides are 1 in., 3 in., and 4 in., respectively.

**Ex. 45.** Construct a triangle whose sides are 2 in., 3 in., and 4 in., respectively. Cut the triangle from the paper and compare it by superposition with the triangles made by other members of your class. What do you conclude must be true about all triangles made according to the directions given ?

## Proposition V. Theorem

**73.** *If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.*




**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$; $BC = EF$; and $AC = DF$.

**Conclusion.** $\triangle ABC \cong \triangle DEF$.

**Proof.** 1. Place $\triangle DEF$ so that $DE$ will coincide with $AB$, $E$ falling on $B$, and so that $F$ falls at $G$, on the opposite side of $AB$ from $C$. Draw $CG$.

2. In $\triangle ACG$, $\angle 1 = \angle 2$, since $AC = AG$.

[In an isosceles $\triangle$, the $\measuredangle$ opposite the equal sides are equal.] § 69

3. In $\triangle BCG$, $\angle 3 = \angle 4$, since $BC = BG$.

4. $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$.

[If equals be added to equals, the sums are equal.] Ax. 3; § 51

5. $\therefore \angle C = \angle G$, or $\angle C = \angle F$. Ax. 7; § 51

6. In $\triangle ABC$ and $\triangle DEF$:
$AC = DF$, and $BC = EF$; Hyp.
$\angle C = \angle F$. Step 5

7. $\therefore \triangle ABC \cong \triangle DEF$.

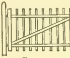
[See Note 1 and § 63.]

**Note 1.** — From now on, an authority which should be familiar to the student will be omitted from demonstrations in the text. The paragraph reference will be given for the present. The student should supply the authority *in full*, without consulting the authority quoted, if possible; otherwise he should look up the reference. When writing out a demon-

stration or giving the demonstration orally, *give all authorities in full* as has been done in the text heretofore.

**Note 2.** — Three sides determine a triangle; that is, the shape and size of the triangle cannot change unless one or more of the sides is changed. Practical use is made of this fact as illustrated in the figures below. In each case, three lengths determine a triangle which makes some part of the object rigid.

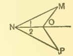
Roof Truss



Gate

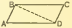
Screen Door

**Ex. 46.** If $MN = NP$ and $MO = OP$, then $NO$ bisects $\angle MNP$.




(Form the Hyp., Con., and give the proof. See § 52.)

**Ex. 47.** If the opposite sides of a quadrilateral $ABCD$ are equal, then $\angle A = \angle C$.

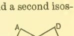



**Ex. 48.** In quadrilateral $ABCD$ in Ex. 47, can you prove that $\angle B = \angle D$?

**Ex. 49.** Why is a shelf bracket made in the form of a triangle?

**Ex. 50.** Can you give any other practical uses of Proposition V?

**Ex. 51.** On segment $XY$ construct isosceles $\triangle XYZ$ and a second isosceles $\triangle XYW$. Draw $ZW$. Prove that $\triangle XZW \cong \triangle YZW$.




**Ex. 52.** In the adjoining figure, if $AB = DC$, and $AC = BD$, then $\angle A$ must equal $\angle D$.

*Suggestion.* — Prove $\triangle ABC \cong \triangle BCD$.

**Note.** — Supplementary Exercises 3 to 5, p. 273, can be studied now.

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## Review Questions

**Ex. 53.** What is a theorem? an axiom?

**Ex. 54.** State three theorems by which two triangles can be proved congruent.

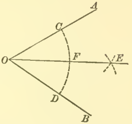
**Ex. 55.** What are homologous parts of congruent triangles?

**Ex. 56.** State Principle I.

**Ex. 57.** What does it mean to bisect an angle?

## Proposition VI. Problem

**74.** *Bisect a given angle.*




**Given** $\angle AOB$.

**Required** to bisect $\angle AOB$.

**Construction.** 1. With $O$ as center and a convenient radius, draw an arc intersecting $AO$ at $C$ and $BO$ at $D$.

2. With $C$ and $D$ as centers and with equal radii, draw arcs intersecting at $E$.

3. Draw $OE$.

**Statement.** $OE$ bisects $\angle AOB$.

**Proof.** The proof is to be given by the pupil.

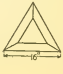
*Suggestions.* — Draw $CE$ and $DE$. Recall § 66.

**Note.** — For construction problems, the regular form is to give the statement of the problem, the parts which are "given," that which is "required," the "construction," the "statement," and the "proof." Also it is important to "discuss" the solution finally, in order to decide when the solution is possible, etc. In this problem it is evident that the solution is always possible.

**Ex. 58.** Draw an obtuse angle. Divide it into four equal parts.

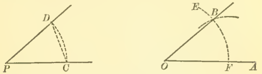
**Ex. 59.** Construct the bisectors of the three angles of a large triangle. What seems to happen ?

**Ex. 60.** Three pieces of wood are to be joined as in the figure on the right. Construct to scale (letting 1″ = 4″) an equilateral triangle ; bisect its angles ; on each bisector lay off a point 4″ from the vertex ; connect these points.



## Proposition VII. Problem

**75.** *At a given point in a line, construct an angle equal to a given angle.*



**Given** $\angle P$, and point $O$ in line $OA$.

**Required** to coustruct an angle equal to $\angle P$, having $O$ as vertex and $OA$ as side.

**Construction.** 1. With $P$ as center and a convenient radius, draw an arc intersecting the sides of $\angle P$ at $C$ and $D$. Draw $CD$.

2. With $O$ as center and $PC$ as radius, draw arc $FE$.

3. With $F$ as center and $CD$ as radius, draw an arc intersecting arc $FE$ at $B$.

4. Draw $OB$.

**Statement.** $\angle AOB = \angle P$.

**Proof.** Draw $FB$. Proof to be given by the pupil.

**Ex. 61.** Construct $\triangle ABC$ with side $AB = 4$ in., and $\angle A$ and $\angle B$ equal to the angles given in the adjoining figure. Measure $AC$. Should the triangles made by different pupils be congruent?


**Ex 62.** Construct $\triangle ABC$ having $\angle A$ equal to the $\angle A$ given in Ex. 61, $AB = 3$ in., and $AC = 2$ in. Measure $BC$.

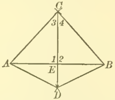
**Note.** — Supplementary Exercises 6–9, p. 273, can be studied now.

**76.** A line perpendicular to a segment at its mid-point is the **Perpendicular-bisector** of the segment.

**Ex. 63.** Draw a line $BC$, 3 in. in length. With compasses, locate a point $A$ above $BC$ which is 2 in. from $B$ and 2 in. from $C$. Locate similarly a point $D$ below $BC$ which is 3 in. from $B$ and 3 in. from $C$. Draw $AD$, cutting $BC$ at $E$. (*a*) Compare $BE$ with $EC$ by means of your dividers. (*b*) Measure $\angle BEA$. (*c*) What kind of lines are $AD$ and $BC$?

## Proposition VIII. Theorem

**77.** *If two points are each equidistant from the ends of a segment, they determine the perpendicular-bisector of the segment.*




**Hypothesis.** $C$ and $D$ are equidistant from the ends of segment $AB$. $CD$ intersects $AB$ at $E$.

**Conclusion.** $AE = EB$; $CD \perp AB$.

**Proof.** 1. In $\triangle ACD$ and $\triangle CDB$:

$AC = CB$, and $AD = DB$; Hyp.

$CD \equiv CD$.

2. $\therefore \triangle ACD \cong \triangle BCD$. Why?

3. $\therefore \angle 3 = \angle 4$. § 65

4. $\triangle ACE \cong \triangle ECB$. § 63

(Give the full proof.)

5. $\therefore AE = EB$. Why?

6. Also $\angle 1 = \angle 2$. Why?

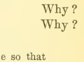
7. $\therefore CD \perp AB$.

[If one straight line meets another straight line so that the adj. $\measuredangle$ formed are equal, the $\measuredangle$ are rt. $\measuredangle$, and the lines are $\perp$.] §§ 26, 29

Note. — It is often necessary, as in this proof, to prove one pair of triangles congruent in order to obtain two equal angles or two equal segments which are required in turn to prove another pair of triangles congruent.

**Ex. 64.** If $XZ$ is the perpendicular-bisector of $AB$, and $Y$ is a point on $XZ$, prove $\triangle AXY \cong \triangle BXY$.

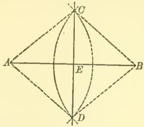
First prove $\triangle AXZ \cong \triangle BXZ$ to get $AX = XB$; then prove $\triangle AZY \cong \triangle BZY$ to get $AY = YB$.




**Note.** — Supplementary Exercises 10–11, p. 274, can be studied now.

## Proposition IX. Problem

**78.** *Construct the perpendicular-bisector of a given segment.*




**Given** line segment $AB$.

**Required** to construct the perpendicular-bisector of $AB$.

**Construction.** 1. With $A$ and $B$ as centers, and with equal radii, draw arcs intersecting at $C$ and also at $D$.

2. Draw $CD$ intersecting $AB$ at $E$.

**Statement.** $E$ bisects $AB$.

**Proof.** 1. $AC = BC$, and also $AD = BD$.

[Radii of equal circles are equal.] § 17

2. $\therefore$ $CD$ is the perpendicular-bisector of $AB$.

[If two points are each equidistant from the ends of a segment, they determine the perpendicular-bisector of the segment.] § 77

**Ex. 65.** Divide a given segment into four equal parts.

**Ex. 66.** Draw a triangle of large size. Construct the perpendicular-bisectors of the three sides. What happens?

**79.** A **Median** of a triangle is the line drawn from a vertex to the mid-point of the opposite side.

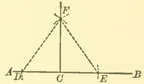
**Ex. 67.** Draw a triangle of large size. Construct the three medians of the $\triangle$. What happens?

**Ex. 68.** Prove that the median drawn to the base of an isosceles triangle bisects the vertical angle. (Construct the figure.)

**Note.** — Supplementary Exercises 12–14, p. 274, can be studied now.

## Proposition X. Problem

**80.** *At a point in a line, construct a perpendicular to the line.*




**Given** $C$, any point in line $AB$.

**Required** to construct a perpendicular to $AB$ at $C$.

**Construction.** 1. With $C$ as center, and any radius, draw arcs intersecting $AB$ at $D$ and $E$ respectively.

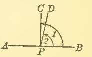
2. With $D$ and $E$ as centers and a radius greater than one half $DE$, draw two arcs intersecting at $F$.

| | | |
|---|---|---|
| 3. | Draw $CF$. | |
| **Statement.** | $CF \perp AB$ at $C$. | |
| **Proof.** 1. | Draw $DF$ and $EF$. | |
| 2. | $C$ is equidistant from $D$ and $E$. | Why? |
| 3. | $F$ is equidistant from $D$ and $E$. | Why? |
| 4. | $\therefore CF$ is the perpendicular-bisector of $DE$. | Why? |
| 5. | $\therefore CF \perp AB$ at $C$. | |

[Since $AB$ and $DE$ are the same straight line.]

**81.** Proposition X proves that *one perpendicular can be drawn to a line at a point in the line.*

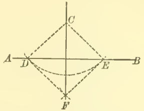
It can be proved that *only one perpendicular can be drawn to a line at a point in the line.* For, if $CP$ and $DP$ were both perpendicular to $AB$ at $P$, then $\angle 1$ and $\angle 2$ would both be right angles and hence would be equal. But $\angle 1$ is greater than $\angle 2$, for the whole is greater than any of its parts.




**Ex. 69.** Prove $CF$ perpendicular to $DE$ (§ 80) by proving that $\angle FCD = \angle FCE$, and then using § 26.

## Proposition XI. Problem

**82.** *Construct a perpendicular to a line from a point not in the line.*



**Given** line $AB$ and point $C$ not in $AB$.

**Required** to construct a $\perp$ to $AB$ from $C$.

**Construction.** 1. With $C$ as center and a convenient radius, draw an arc intersecting $AB$ at $D$ and $E$ respectively.

2. With $D$ and $E$ as centers, and equal radii, draw two arcs intersecting at F.

3. Draw $CF$.

**Statement.** $CF \perp AB$.

| | | |
|---|---|---|
| **Proof.** 1. | $C$ is equidistant from $D$ and $E$. | Why? |
| 2. | $F$ is equidistant from $D$ and $E$. | Why? |
| 3. | $\therefore CF$ is the perpendicular-bisector of $DE$. | Why? |
| 4. | $\therefore CF \perp AB$. | |

[Since $AB$ and $DE$ are the same straight line.]

**Historical Note.** — This construction is attributed to Oenipodes of Chios (465 B.C.)

**83.** Proposition XI proves that *one perpendicular can be drawn to a line from a point not in the line.*

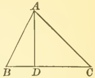
It will be proved later (§ 88) that *only one perpendicular can be drawn to a line from a point not in the line.*

It will also be proved (§ 164) that *the perpendicular is the shortest segment from the point to the line.*

**Note.** — Supplementary Exercises 15–16, p. 274, can be studied now.

**84.** The **Distance** from a point to a line is the length of the perpendicular from the point to the line.

**85.** An **Altitude** of a triangle is the perpendicular drawn from a vertex to the opposite side or the opposite side extended; as, $AD$.

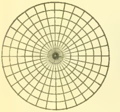
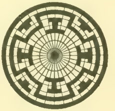
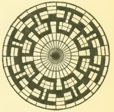



**Ex. 70.** How many altitudes does a triangle have?

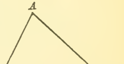
**Ex. 71.** Construct a triangle whose sides are 2 inches, 3 inches, and 4 inches, respectively. Construct the three altitudes of the triangle.

**Ex. 72.** Construct an angle of 45°. (Use Prop. XI and Prop. VI.) Construct an angle of 135°; of $22\frac{1}{2}°$; of $67\frac{1}{2}°$.

**Note.** — The second and third designs below are drawn upon the first figure as a background. Can you discover how the first figure is constructed? Can you make an original design similar to these designs?

**86.** An **Exterior Angle** of a triangle is the angle at any vertex formed by a side of the triangle and the adjacent side extended; as, $\angle DCA$.



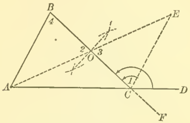

One interior angle is *adjacent* to the exterior angle and the other two are *remote interior* angles. Thus, $\angle A$ and $\angle B$ are the remote interior angles of exterior $\angle DCA$.

**Ex. 73.** Draw a large figure like that in § 86. Measure the exterior $\angle DCA$ and each of the remote interior angles. How does the exterior angle compare with the remote interior angles?

**Ex. 74.** In the figure for Prop. IX, p. 44, $\angle CEB$ is an exterior angle of what triangles?

## Proposition XII. Theorem

**87.** *An exterior angle of a triangle is greater than either remote interior angle.*



**Hypothesis.** $\angle BCD$ is an exterior $\angle$ of $\triangle ABC$.

**Conclusion.** $\angle BCD > \angle B$; also $\angle BCD > \angle A$.

Part I. **Proof.** 1. Through $O$, the mid-point of $BC$, draw $AO$. Extend $AO$ to $E$, making $OE$ equal to $AO$. Draw $CE$.

2. $\therefore \triangle ABO \cong \triangle OCE$.

[Give the full proof.]

3. $\therefore \angle 4 = \angle 1$. Why?

4. $\angle BCD > \angle 1$.

[The whole is greater than any of its parts.] Ax. 8, § 51

5. $\therefore \angle BCD > \angle 4$.

(Substitute $\angle 4$ for its equal, $\angle 1$.) Ax. 2, § 51

Part II. **Proof.** 1. Extend $BC$ to $F$.

2. $\angle ACF > \angle A$.

[By a proof similar to that for Part I.]

3. $\angle BCD = \angle ACF$. Why?

4. $\therefore \angle BCD > \angle A$.

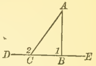
(Substitute $\angle BCD$ for its equal, $\angle ACF$, in step 2.)

**Ex. 75.** In the figure for Prop. XII, prove that $\angle BOE$ is greater than $\angle 4$. (Use § 87.)

**Ex. 76.** Prove also that $\angle AOC > \angle 1$.

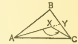
**Ex. 77.** Compare $\angle ECF$ with $\angle 3$.

**88. Cor.** *There can be only one perpendicular from a point to a line.*




If $AB \perp DE$, then $AC$ cannot be $\perp DE$, for $\angle 2 > \angle 1$ and hence $\angle 2$ is an obtuse angle.

**Ex. 78.** If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides. (Prove $\angle AXC > \angle ABC$.)




*Suggestions.*—1. Compare $\angle AXC$ with $\angle AYC$. 2. Compare $\angle AYC$ with $\angle ABC$.

### Review Questions

**Ex. 79.** What is the perpendicular-bisector of a segment?

**Ex. 80.** What is a median of a triangle? How many medians does a triangle have?

**Ex. 81.** What is an altitude of a triangle?

**Ex. 82.** What is the distance from a point to a line?

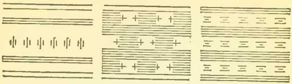
**Ex. 83.** (*a*) What is an exterior angle of a triangle? (*b*) How many exterior angles can be formed at one vertex of a triangle? (*c*) How do they compare?

**Ex. 84.** (*a*) How many perpendiculars can be drawn to a line from a point not in the line? (*b*) How many at a point of the line?

**Ex. 85.** Five fundamental construction problems have now been taught (§§ 74, 75, 78, 80, 82). (*a*) State each of them. (*b*) Are you able to make each of these constructions quickly with ruler and compass alone?

**Note.** — Straightedge and compass alone are employed in making the constructions in elementary geometry. This practice was initiated by Plato (429–348 B.C.). Naturally some constructions cannot be made with these tools alone. For example, *it is impossible to trisect an angle by ruler and compass alone.*

## PARALLEL LINES

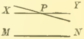


SOME PARALLEL LINE BORDER DESIGNS

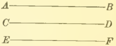
**89.** Two straight lines are **Parallel** ($\parallel$) if they lie in the same plane and do not meet however far they are extended.

**Note.** — Two straight lines in the same plane either intersect or are parallel lines.

**90. Ax. 16. Axiom of Parallels.** *Through a given point there can be only one parallel to a given line.* Thus, $XY \parallel MN$.




**91. Cor.** *If two lines are parallel to a third line, they are parallel to each other.*

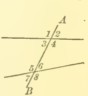



**Hyp.** $AB \parallel CD$; $EF \parallel CD$.

**Con.** $AB \parallel EF$.

**Proof.** If $AB$ and $EF$ are not parallel, they must meet at a point. Through this point there would then be two lines parallel to $CD$. But this is impossible by the Axiom of Parallels. Hence $AB$ must be parallel to $EF$.

**92.** If two lines are cut by a third line, $AB$, called a **Transversal**, the angles are named as follows:




$\angle$s 3, 4, 5, and 6 are called **Interior Angles**.

$\angle$s 1, 2, 7, and 8 are called **Exterior Angles**.

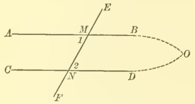
$\angle$s 3 and 6 are called **Alternate-interior Angles**; also $\angle$s 5 and 4.

$\angle$s 1 and 8 are called **Alternate-exterior Angles**; also $\angle$s 2 and 7.

$\angle$s 2 and 6 are called **Corresponding Angles**; also $\angle$s 1 and 5, $\angle$s 3 and 7, and $\angle$s 4 and 8.

## Proposition XIII. Theorem

**93.** *If two lines are cut by a transversal so that a pair of alternate-interior angles are equal, the lines are parallel.*




**Hypothesis.** $AB$ and $CD$ are cut by $EF$; $\angle 1 = \angle 2$.

**Conclusion.** $AB \parallel CD$.

**Proof.** 1. Suppose that $AB$ is not parallel to $CD$, and that it meets $CD$ at point $O$ on the right of $EF$, forming $\triangle MNO$.

2. Then $\angle 1$ is an exterior angle of $\triangle MNO$.

3. $\therefore \angle 1 > \angle 2$

[An ext. $\angle$ of a $\triangle$ is greater than either remote int. $\angle$.] § 87

4. But $\angle 1 = \angle 2$ Hyp.

5. $\therefore AB$ cannot meet $CD$ on the right of $EF$.

6. Similarly it can be proved that $AB$ cannot meet $CD$ on the left of $EF$.

7. $\therefore AB \parallel CD$

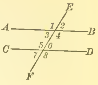
[Two lines are $\parallel$ if they lie in the same plane and do not meet however far they are extended.] § 89

**Note.** — Read the next paragraph at this time.

**94.** The method of proof used in §§ 91 and 93 is called the **Indirect Method of Proof.** Notice: (*a*) it starts by assuming the negative of the conclusion; (*b*) it follows up the consequences of this assumption until a statement is reached which contradicts a known fact; (*c*) this contradiction is made the basis for asserting that the desired conclusion is true.

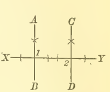
**95. Principle II.** To prove two lines parallel, try to prove a pair of alternate-interior angles equal.

**96. Cor. 1.** *If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.*



**Hyp.** $AB$ and $CD$ are cut by $EF$; $\angle 2 = \angle 6$.
**Con.** $AB \parallel CD$.
**Plan.** Try to prove $\angle 3 = \angle 6$. Then use § 93.

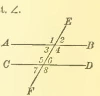
**97. Cor. 2.** *If two lines are perpendicular to a third line, they are parallel.*



**Hyp.** $AB \perp XY$; $CD \perp XY$.
**Con.** $AB \parallel CD$.
**Plan.** Try to prove $\angle 1 = \angle 2$. Then use § 93.

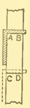
**98. Cor. 3.** *If two lines are cut by a transversal so that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.*

**Hyp.** $AB$ and $CD$ are cut by $EF$; $\angle 4 + \angle 6 = 1$ st. $\angle$.
**Con.** $AB \parallel CD$.
**Plan.** Try to prove $\angle 3 = \angle 6$.



| | | |
|---|---|---|
| **Proof.** 1. | $\angle 3 + \angle 4 = 1$ st. $\angle$. | § 39 |
| 2. | $\angle 4 + \angle 6 = 1$ st. $\angle$. | Hyp. |
| 3. | $\therefore \angle 3 + \angle 4 = \angle 4 + \angle 6$. | Why? |
| 4. | $\therefore \angle 3 = \angle 6$, and hence $AB \parallel CD$. | Why? |

**Ex. 86.** If two lines are cut by a transversal so that a pair of alternate-exterior angles are equal, the lines are parallel.




**Ex. 87.** A carpenter wants a board of length $AC$ with parallel ends, $AB$ and $CD$. He marks $AB$ and $CD$ by means of his square as in the figure. Why must $AB$ and $CD$ be parallel? Assume that edge $AC$ is a straightedge.

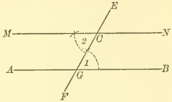
**Ex. 88.** If sides $BA$ and $CA$ of any $\triangle ABC$ are extended their own lengths through vertex $A$ to $D$ and $E$ respectively, then $DE$ is parallel to $BC$.

*Suggestion.* — Apply § 95 and § 66.

**Note.** — Supplementary Exercises 17–21, p. 274, can be studied now.

## Proposition XIV. Problem

**99.** *Construct a parallel to a line through a point not in the line.*




**Given** line $AB$ and $C$, any point not in line $AB$.

**Required** to construct a parallel to $AB$ through $C$.

**Construction** ($a$). 1. Draw $FE$ through $C$, meeting $AB$ at $G$.

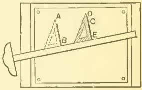
2. At $C$, construct $\angle 2 = \angle 1$.

**Statement.** $MN \parallel AB$. Why?

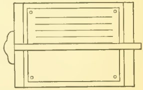
**Proof.** To be given by the pupil.

**Construction** ($b$). A second construction is based upon § 96. This construction is left as an exercise for the pupil.

**Ex. 89.** The figure adjoining shows how a draughtsman draws a line through $C$ parallel to $AB$. Why is $DE$ parallel to $AB$?




**Ex. 90.** The adjoining figure shows how a draughtsman draws parallel lines by means of his T-square. Why are the lines parallel?

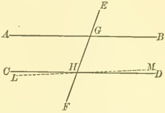


**Ex. 91.** Construct the figure for § 97; then construct the bisectors of $\angle$s 1 and 2. Prove that these bisectors are parallel.

**Ex. 92.** In the figure for Prop. XII, p. 48, prove that $CE$ is parallel to $AB$.

## Proposition XV. Theorem

**100.** *If two parallels are cut by a transversal, alternate-interior angles are equal.*




**Hypothesis.** $EF$ cuts ∥s $AB$ and $CD$ at $G$ and $H$.

**Conclusion.** $\angle AGH = \angle GHD$.

**Proof.** 1. Suppose that $\angle AGH$ is less than $\angle GHD$.

2. Then draw $LM$ through $H$, so that $\angle GHM = \angle AGH$.

3. $\therefore LM \parallel AB$.

[If two lines are cut by a transversal so that a pair of alt.-int. ∠s are equal, the lines are ∥.] § 93

4. But this is impossible, for $CD \parallel AB$, by hypothesis.

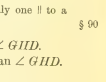
[Through a given point, there can be only one ∥ to a given line.] § 90

5. $\therefore \angle AGH$ cannot be less than $\angle GHD$.

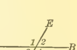
6. Similarly, $\angle AGH$ cannot be greater than $\angle GHD$.

7. $\therefore \angle AGH = \angle GHD$.

**Note.** — Review § 94.

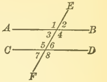



**Ex. 93.** If $EF$ cuts parallels $AB$ and $CD$ so that $\angle 3 = 30°$, how many degrees are there in each of the other angles of the figure?




**Ex. 94.** If $EF$, joining two parallels, be bisected and $GH$ be drawn through the mid-point and included between the parallels, then $GH$ will also be bisected by the point.

**101.** **Cor.** 1. *If two parallels are cut by a transversal, corresponding angles are equal.*

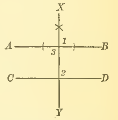



**Hyp.** $AB \parallel CD$; $\angle 2$ and $\angle 6$ are corresponding angles.

**Con.** $\angle 2 = \angle 6$.

[What do you know about $\angle 3$ and $\angle 6$ ?]

**102.** **Cor.** 2. *If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*




**Hyp.** $AB \parallel CD$ ; $XY \perp AB$.

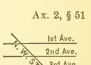
**Con.** $XY \perp CD$.

[What must be proved about $\angle 2$ ?]

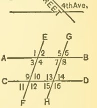
**103.** **Cor.** 3. *If two parallels are cut by a transversal, interior angles on the same side of the transversal are supplementary.*

**Hyp.** $AB \parallel CD$; $\angle 4$ and $\angle 6$ are int. $\angle$s on the same side of the transversal. (Fig. § 101.)

| | | |
|---|---|---|
| **Con.** | $\angle 4 + \angle 6 = 1$ st. $\angle$. | |
| **Proof.** 1. | $\angle 4 + \angle 3 = 1$ st. $\angle$. | § 39 |
| 2. | $\angle 6 = \angle 3$. | Why? |
| 3. | $\therefore \angle 4 + \angle 6 = 1$ st. $\angle$. | |
| | [Substituting for $\angle 3$ its equal, $\angle 6$.] | Ax. 2, § 51 |




**Ex. 95.** If N. W. Street crosses 1st Ave. at an angle of 45°, at what angle does it cross the parallel avenues?



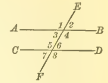

**Ex. 96.** In the adjoining figure, if $AB \parallel CD$, and $EF \parallel GH$, prove: (*a*) $\angle 1 = \angle 13$;

(*b*) $\angle 3 + \angle 16 = 1$ st. $\angle$.

**Ex. 97.** If a line be drawn parallel to the base of an isosceles triangle, cutting the two sides of the triangle, it makes equal angles with these sides.

**104.** One theorem is called the **Converse** of another when the hypothesis and conclusion of the one become the conclusion and hypothesis of the other. Thus, Prop. XV is the converse of Prop. XIII.

| In Proposition XIII | In Proposition XV |
|---|---|
| **Hyp.** $EF$ cuts $AB$ and $CD$. $\angle 3 = \angle 6$. | **Hyp.** $EF$ cuts $AB$ and $CD$. $AB \parallel CD$. |
| **Con.** $AB \parallel CD$. | **Con.** $\angle 3 = \angle 6$. |




The converse of a given theorem is not always true.

Thus, *all right angles are equal.* The converse would be *all equal angles are right angles.* Evidently this is not true.

**Ex. 98.** Of what statement is Cor. 1 (§ 101) the converse? Cor. 2 (§ 102)? Cor. 3 (§ 103)?

**Ex. 99.** In the figure for § 101, prove $\angle 3 = \angle 7$.

**Ex. 100.** In the figure for § 101, prove $\angle 3 + \angle 5 =$ 1st $\angle$.

**Ex. 101.** If two parallels are cut by a transversal, alternate exterior angles are equal.

**Ex. 102.** State the converse of Ex. 101. Is it a true statement?

**Ex. 103.** If two parallels are cut by a transversal, bisectors of a pair of corresponding angles are parallel.

**Ex. 104.** If two parallels are cut by a transversal, exterior angles on the same side of the transversal are supplementary.

*Suggestion.* — The proof is like that for § 103.

**Ex. 105.** State the converse of Ex. 104.

**Ex. 106.** If two lines are perpendicular to parallel lines, then they are either parallel or coincide.

**Ex. 107.** What is the axiom of parallels?

**Ex. 108.** What are parallel lines?

**Ex. 109.** Is the following a correct statement? When two lines are cut by a transversal, alternate interior angles are equal.

**Note.** — The various theorems about angles made by a transversal of two parallels are found in Euclid. They were probably formulated by the Pythagoreans.

## Proposition XVI. Theorem

**105.** *If two angles have their sides respectively parallel, they are equal, provided both pairs of parallels extend in the same directions from their vertices, or in opposite directions.*

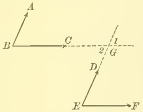


Fig. 1

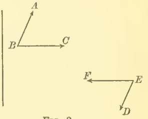


Fig. 2

I. (Fig. 1.) **Hypothesis.** $\triangle ABC$ and $DEF$ have $AB \parallel DE$ and $BC \parallel EF$.

**Conclusion.** $\angle B = \angle E.$

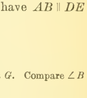
[Proof to be given by the pupil.]

*Suggestion.* — Extend $BC$ and $ED$ until they intersect at $G$. Compare $\angle B$ with $\angle 2$ and $\angle E$ with $\angle 2$. Then compare $\angle B$ with $\angle E$.

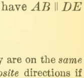
II. (Fig. 2.) **Hypothesis.** $\triangle ABC$ and $DEF$ have $AB \parallel DE$ and $BC \parallel EF$.

**Conclusion.** $\angle B = \angle E.$

**Note.** — The sides extend in the *same* direction if they are on the *same* side of a straight line joining their vertices, and in *opposite* directions if they are on *opposite* sides of this line.



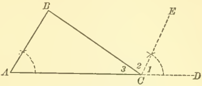

**Ex. 110.** If two angles have their sides respectively parallel, one pair of parallels extending in the same directions but the other pair extending in opposite directions from their vertices, the angles are supplementary. (Prove $\angle B + \angle E = 1$ st. $\angle$.)




**Ex. 111.** Two streets cross as in the adjoining figure. If the lot lines at corner $C$ make an angle of 70°, determine the number of degrees in the angle formed at each of the other corners.

## Proposition XVII. Theorem

**106.** *The sum of the angles of any triangle is one straight angle.*




**Hypothesis.** $\triangle ABC$ is any triangle.

**Conclusion.** $\angle A + \angle B + \angle C = 1$ st. $\angle$.

**Proof.** 1. Extend $AC$ to $D$, and construct $CE$ parallel to $AB$.

2. $\angle 1 + \angle 2 + \angle 3 = 1$ st. $\angle$.

[The sum of all the successive adj. $\angle$s around a point on one side of a st. line is one st. $\angle$.] § 34

3. $\angle A = \angle 1$. Const.

4. $\angle B = \angle 2$, since $BC$ cuts $\parallel$s $AB$ and $CE$. Why?

5. $\therefore \angle A + \angle B + \angle C = 1$ st. $\angle$.

[Substituting $\angle A$ for $\angle 1$, $\angle B$ for $\angle 2$, and $\angle C$ for $\angle 3$.] Ax. 2, § 51

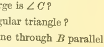
**Note.** — This theorem is attributed to Eudemus, a pupil of Aristotle.

**Ex. 112.** If $\angle A = 70°$, and $\angle B = 35°$, how large is $\angle C$?

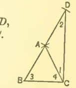
**Ex. 113.** How large is each angle of an equiangular triangle?

**Ex. 114.** Prove Prop. XVII by constructing a line through $B$ parallel to $AC$.

**Ex. 115.** The rafters of a "saddle roof" make an angle of 40° with a level line. What angle do the rafters form at the ridge?




**Ex. 116.** $AB = AC$ in $\triangle ABC$. $BA$ is extended to $D$, so that $AD = AB$. Prove that $CD$ is perpendicular to $BC$.




*Suggestions.* — 1. $\angle 1 + \angle 4$ must be proved a right $\angle$.
2. What part of $\angle 1 + \angle 2 + \angle 3 + \angle 4$ is $\angle 1 + \angle 4$?

**Note.** — This is a very important exercise. It may be expressed thus: if the median to one side of a triangle is one half of that side, the angle from which it is drawn is a right angle.

**107.** A triangle is a **Right Triangle** when it has one right angle.

The **Hypotenuse** of a right triangle is the side opposite the right angle; the **Legs** of a right triangle are the two sides of the triangle including the right angle.

If the legs of a right triangle are equal, the triangle is called an **Isosceles Right Triangle.**

### Corollaries to Proposition XVII

**108. Cor.** 1. *A triangle cannot have two right angles or two obtuse angles.*

**109. Cor.** 2. *The acute angles of a right triangle are complementary.*

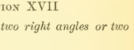
**110. Cor.** 3. *An exterior angle of a triangle equals the sum of the two remote interior angles.*




Prove $\angle 1 = \angle 3 + \angle 4$.

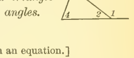
[$\angle 2 + \angle 1 =$ ? $\angle 2 + \angle 3 + \angle 4 =$ ? Form an equation.]

**111. Cor.** 4. *If two angles of one triangle equal respectively two angles of another triangle, the third angles are equal.*




**Hyp.** $\angle 1 = \angle 4$, and $\angle 2 = \angle 5$.
**Con.** $\angle 3 = \angle 6$.

**112. Cor.** 5. *If two triangles have a side, the opposite angle, and another angle of the one equal respectively to a side, the opposite angle, and another angle of the other, the triangles are congruent.*



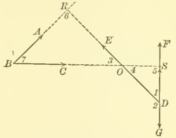

**Hyp.** $AB = DE$; $\angle C = \angle F$; $\angle B = \angle E$.
**Con.** $\triangle ABC \cong \triangle DEF$.

*Suggestions.*—1. Prove $\angle A = \angle D$.
2. Then prove $\triangle ABC \cong \triangle DEF$ by § 63.

**Note.**—Supplementary Exercises 22–35, p. 275, can be studied now.

## Proposition XVIII. Theorem

**113**. *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*




**Hypothesis.** $\measuredangle ABC$ and $EDF$ have $AB \perp DE$ and $BC \perp FG$.

**Conclusion.** $\angle 7 = \angle 1.$
$\angle 7 + \angle 2 = 1$ st. $\angle$.

**Proof.** 1. Extend $DE$ until it meets $BA$ at $R$. Extend $BC$ until it meets $FG$ at $S$ and intersects $DR$ at $O$.

| | | |
|---|---|---|
| 2. | In $\triangle RBO$ and $\triangle ODS$: | |
| | $\angle 6 = \angle 5;$ | Why? |
| | $\angle 3 = \angle 4.$ | Why? |
| 3. | $\therefore \angle 7 = \angle 1.$ | § 111 |
| 4. | $\angle 1 + \angle 2 = 1$ st. $\angle$. | Why? |
| 5. | $\therefore \angle 7 + \angle 2 = 1$ st. $\angle$. | |
| | [Substitute $\angle 7$ for its equal $\angle 1$.] | Ax. 2, § 51 |

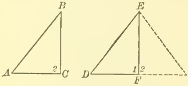
**Note.** — The angles are equal if both are acute or both obtuse. They are supplementary if one is acute and one is obtuse.

**Ex. 117.** If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, they are congruent. (§ 112.)

**Ex. 118.** If two right triangles have a leg and the opposite acute angle of one equal respectively to a leg and the opposite acute angle of the other, they are congruent.

## Proposition XIX. Theorem

**114.** *If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.*



**Hypothesis.** In rt. $\triangle$ $ABC$ and $DEF$:

hypotenuse $AB =$ hypotenuse $DE$; $BC = EF$.

**Conclusion.** $\triangle ABC \cong \triangle DEF$.

**Proof.** 1. Place $\triangle ABC$ beside $\triangle DEF$ so that $BC$ will coincide with its equal $EF$, $B$ falling on $E$, and so that $A$ falls at $G$, on the opposite side of $EF$ from $D$.

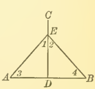
| | | |
|---|---|---|
| 2. | $\therefore \angle 1 + \angle 2 = 1$ st. $\angle$. | Why? |
| 3. | $\therefore$ $DFG$ is a straight line. | § 40 |
| 4. | $\therefore$ figure $EDFG$ is a triangle. | |
| 5. | $\therefore \angle G = \angle D$, or $\angle A = \angle D$. | § 69 |
| 6. | In $\triangle ABC$ and $\triangle DEF$: | |
| | $AB = DE$; $\angle 1 = \angle 2$; $\angle A = \angle D$. | |
| 7. | $\therefore \triangle ABC \cong \triangle DEF$. | § 112 |

**115. Cor.** *If two equal oblique segments are drawn to a line from a point in a perpendicular to the line:*

(1) *they cut off equal distances from the foot of the perpendicular.* (Prove $AD = DB$.)

(2) *they make equal angles with the perpendicular.* (Prove $\angle 1 = \angle 2$.)

(3) *they make equal angles with the given line.* (Prove $\angle 3 = \angle 4$.)



**Note.** — Supplementary Exercise 36, p. 276, can be studied now.

## SUMMARY

**116.** The student will have constant use for the foregoing theorems, problems, and facts.

A. **Two triangles are congruent if :**

1. Two sides and the included ∠ of one are equal respectively, etc. § 63
2. Two ≰ and the included side of one are equal respectively, etc. § 67
3. The three sides of one are equal respectively, etc. § 73
4. A side, the opposite ∠, and another ∠ of one are equal respectively, etc. § 112

B. **Two right triangles are congruent if :**

1. The hypotenuse and a leg of one are equal respectively, etc. § 114
2. The hypotenuse and an acute ∠ of one are equal respectively, etc. Ex. 117
3. A leg and the opposite acute ∠ of one are equal respectively, etc. Ex. 118

C. **Two lines are parallel if :**

1. Alt.-int. ≰ made by a transversal are equal. § 93
2. Corresponding ≰ made by a transversal are equal. § 96
3. Int. ≰ on the same side of the transversal are supp. § 98
4. They are parallel to, or perpendicular to, the same line. §§ 91, 97

D. **If a transversal cuts two parallels :**

1. Alt.-int. ≰ are equal. § 100
2. Corresponding ≰ are equal. § 101
3. Int. ≰ on the same side of the transversal are supp. § 103

E. **To prove two line segments are equal :**

Try to prove them homologous sides of congruent ▲. § 66

F. **To prove two angles equal, try to prove that they :**

1. Are homologous ≰ of congruent ▲. § 66
2. Are supplements or complements of the same or equal ≰. §§ 37, 41
3. Are right ≰ or vertical ≰. §§ 27, 54
4. Are opposite the equal sides of an isosceles △. § 69
5. Are alt.-int. ≰ or corresponding ≰ made by a transversal of two parallels. §§ 100, 101
6. Have their sides respectively ∥ or ⊥, etc. §§ 105, 113

G. **To prove an angle is a right angle, try to prove :**

1. It is equal to its supplement. § 26
2. It is the angle formed by two lines which are ⊥ by § 77.
3. It is equal to an angle known to be a right ∠.

**117.** Success in **demonstrating unproved theorems** comes as a result of knowledge of the facts summarized in § 116, systematic methods of studying a theorem and planning its demonstration, and experience and perseverance.

DIRECTIONS

1. Read the theorem carefully, making certain that each word is thoroughly understood.

2. Draw the figure carefully, constructing it when possible.

Make the figure general. Thus, if the figure is based upon a triangle, do not draw a right triangle or an isosceles triangle unless told to do so.

3. Decide upon the hypothesis and conclusion.

(*a*) Remember that the hypothesis states the facts about the figure which are assumed, and that the conclusion states the facts which are to be proved.

(*b*) If the theorem is stated in the "if . . . then . . ." form (§ 52), the hypothesis and conclusion are evident at once.

(*c*) If the theorem is not stated in the "if . . . then . . ." form, the declarative sentence in its simplest form will give the conclusion, and the subject of the sentence with its modifiers will give the hypothesis.

4. Decide upon a **plan** for the demonstration.

For the present, the suggestions in § 116 will aid the pupil in solving most exercises.

Ask "what does the conclusion mean?" or "how can I prove the conclusion?" The answers will suggest a plan for the proof.

Thus, suppose that the conclusion is: $EA = EB$.

*Question.* How can I prove $EA = EB$?

*Answer.* By proving them homologous parts of congruent ▵.

This means that two triangles of which $EA$ and $EB$ are sides must be selected.

*Question.* How can I prove two triangles congruent?

*Answer.* By one of the methods given in § 116, A and B.

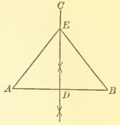
This leads to the comparison of the sides and angles of the triangles.

*Question.* What do I know about the sides and ∠ of the ▵? How can I prove these two angles equal?

*Answer.* § 116, E and F suggest possible answers.

## Proposition XX. Theorem

**118.** I. *Any point in the perpendicular-bisector of a segment is equidistant from the ends of the segment.*



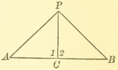

**Hypothesis.** $CD \perp AB$; $AD = DB$; $E$ is any point in $CD$.

**Conclusion.** $EA = EB$.

**Plan.** Try to prove $EA$ and $EB$ hom. sides of cong. $\triangle$s.

[Proof to be given by the pupil.]

II. (Converse.) *Any point equidistant from the ends of a segment lies in the perpendicular-bisector of the segment.*




**Hypothesis.** $AB$ is a st. line.

$PA = PB$.

**Conclusion.** $P$ lies in the perpendicular-bisector of $AB$.

**Plan.** Let $C$ be the mid-point of $AB$. Try to prove $PC \perp AB$, by proving $\angle 1 = \angle 2$.

**Proof.** 1. $\triangle APC \cong \triangle PCB$.

[Give the full proof.]

2. $\therefore \angle 1 = \angle 2$. Why?

3. $\therefore PC \perp AB$.

[If one st. line meets another st. line so that the adj. $\angle$s formed are equal, the $\angle$s are rt. $\angle$s and the lines are $\perp$.]

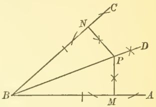
§§ 26, 29

**119. Cor.** *Two obliques, drawn to a line from a point in a perpendicular to the line and cutting off equal distances from the foot of the perpendicular, are equal.*

**Note.** — Supplementary Exercises 37–38, p. 276, can be studied now.

## Proposition XXI. Theorem

**120.** I. *Any point in the bisector of an angle is equidistant from the sides of the angle.*



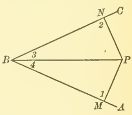

**Hypothesis.** $BD$ bisects $\angle ABC$; $P$ is in $BD$; $PM \perp AB$; $PN \perp BC$.

**Conclusion.** $PM = PN$.

**Plan.** Try to prove $PM$ and $PN$ hom. parts of cong. $\triangle$s.

*Suggestion.* — Recall § 112.

II. (Converse.) *Any point equidistant from the sides of an angle lies in the bisector of the angle.*




**Hypothesis.** $P$ lies within $\angle ABC$; $PM \perp AB$; $PN \perp BC$; $PM = PN$.

**Conclusion.** $P$ lies in the bisector of $\angle ABC$.

**Plan.** Draw $PB$. Try to prove $PB$ bisects $\angle ABC$, by proving $\angle 3 = \angle 4$.

**121. Cor.** *Any point not in the bisector of an angle is unequally distant from the sides of the angle.*

**Note.** — Supplementary Exercise 39, p. 276, can be studied now.

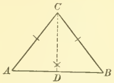
### ISOSCELES AND EQUILATERAL TRIANGLES

**122.** Review the definitions of isosceles and equilateral triangles in § 68; also review Prop. III, § 70, and Ex. 113.

Notice that an equilateral triangle is a special form of isosceles triangle. Hence, for each theorem about an isosceles triangle there is a corresponding theorem about an equilateral triangle, which may be considered a corollary of the former. Thus, § 70 follows at once from Prop. III.

## Proposition XXII. Theorem

**123.** *If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*




**Hypothesis.** In $\triangle ABC$, $\angle A = \angle B$.

**Conclusion.** $AC = BC$.

**Plan.** Try to prove $AC$ and $BC$ hom. sides of cong. $\triangle$s.

**Proof.** 1. Construct $CD$ bisecting $\angle ACB$.

[Complete the proof in good form.]

**124. Cor.** *If a triangle is equiangular, it is also equilateral.*

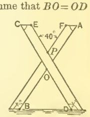
**Ex. 119.** Prove that the bisector of the vertical angle of an isosceles triangle is perpendicular to and bisects the base.

**Ex. 120.** Prove that the altitude to the base of an isosceles triangle is also the median to the base and bisects the vertical angle.

**Ex. 121.** Prove that the altitudes drawn to the equal sides of an isosceles triangle are equal.

**Ex. 122.** Prove that the medians drawn to the equal sides of an isosceles triangle are equal.

**Ex. 123.** A boy wishes to make a saw-buck. Assume that $BO = OD$ and that $\angle EPF = 40°$. Determine the angles at $B$ and $D$ so that the pieces $AB$ and $CD$ will stand firmly upon the ground. Determine the angles at $C$ and $A$ so that $CE$ and $FA$ will be parallel to the ground line.

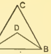



**Ex. 124.** Construct an angle of 60°.

(Construct an equilateral triangle.)

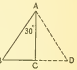
Also construct an angle of : 30°; 120°; 150°.

**Ex. 125.** If one angle of an isosceles triangle is 60°, the triangle is equilateral.




**Ex. 126.** The bisectors of the equal angles of an isosceles triangle form with the base another isosceles triangle.

**Ex. 127.** If the bisector of the exterior angle at one vertex of a triangle is parallel to the side joining the other two vertices, the triangle is isosceles.

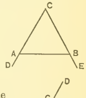



**Ex. 128.** If one acute angle of a right triangle is 30°, the side opposite is one-half the hypotenuse.

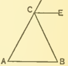
*Suggestion.* — Extend $BC$ to $D$, making $CD$ equal to $BC$. Prove $\triangle ABD$ is equilateral.

Note. — This is a very important exercise. Pupils should endeavor to remember it, as it will be required in the proofs of certain exercises in geometry.

**Ex. 129.** Prove that the perpendiculars drawn from the mid-point of the base of an isosceles triangle to the sides of the triangle are equal.

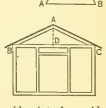



**Ex. 130.** If the equal sides of an isosceles triangle be extended beyond the base, the exterior angles so formed are equal.




**Ex. 131.** Prove that the bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base of the triangle.

*Suggestion.* — Compare $\angle BCD$ with $\angle A + \angle B$ (§ 110).




**Ex. 132.** In the gable in the front of a garage, the two boards whose upper edges are $AB$ and $AC$ are of equal length and meet at a point $A$ on a line $AD$ which is perpendicular to $BC$.

If $\angle ACD = 30°$, how large are $\angle ABD$, $\angle CAD$, and $\angle BAD$?

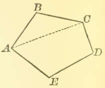
**Ex. 133.** If the perpendiculars drawn from the mid-point of one side of a triangle to the other two sides are equal, the triangle is isosceles.

**Ex. 134.** If the altitudes drawn to two sides of a triangle are equal, the triangle is isosceles.

**Note.** — Supplementary Exercises 40–49, p. 276, can be studied now.

## POLYGONS

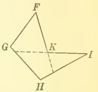
**125.** A **Polygon** is a *closed* (§ 6) broken line; as $ABCDE$.




Points $A$, $B$, $C$, etc., are the *vertices* of the polygon; $\angle s\ A$, $B$, $C$, etc., are the *angles*; $AB$, $BC$, $CD$, etc., are the *sides*; the sum of the lengths of the sides is the *perimeter* of the polygon; a line joining any two non-consecutive vertices is a *diagonal* of the polygon; as $AC$.

A polygon incloses a portion of the plane called the **interior of the polygon**.

**126.** A polygon is **Convex** if no side, when extended, will pass through the interior of the polygon; as $ABCDE$ of § 125.




A polygon is **Concave** if at least two sides, when extended, will pass through the interior of the polygon; as $FGHIK$.

**127.** Only convex polygons are considered in this text. A convex polygon having $n$ sides has $n$ vertices.

**128.** An **Equilateral Polygon** is one whose sides are all equal. An **Equiangular Polygon** is one whose angles are all equal.

**129.** Two polygons are **mutually equilateral** if the sides of one are equal respectively to the sides of the other; and **mutually equiangular** if the angles of one are equal respectively to the angles of the other. If two polygons are both mutually equiangular and mutually equilateral, they are congruent.

**130.** The principal polygons are named as follows:

| No. of Sides | Name of the Polygon | No. of Sides | Name of the Polygon |
|---|---|---|---|
| 3 | Triangle | 7 | Heptagon |
| 4 | Quadrilateral | 8 | Octagon |
| 5 | Pentagon | 10 | Decagon |
| 6 | Hexagon | $n$ | $n$-gon |

## QUADRILATERALS

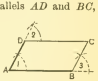
**131.** A **Parallelogram** ($\square$) is a quadrilateral whose opposite sides are parallel.

A pair of parallel sides are called *bases;* the perpendicular distance between them is called the *altitude.*

**132. Cor.** *Two consecutive angles of a parallelogram are supplementary.*

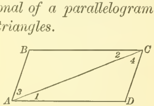
For, in the figure of § 131, since $AB$ cuts the parallels $AD$ and $BC$, $\angle A + \angle B = 1 \text{ st. } \angle$. (§ 103.)

**Ex. 135.** Construct a $\square$ $ABCD$, making $AD = 2$ in., $AB = 3$ in., and $B = 60°$. After you have constructed the figure, compare the opposite sides by means of your dividers.



## Proposition XXIII. Theorem

**133.** *A diagonal of a parallelogram divides it into two congruent triangles.*



**Hypothesis.** $ABCD$ is a parallelogram. $AC$ is a diagonal.

**Conclusion.** $\triangle ABC \cong \triangle ACD$.

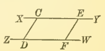
[Proof to be given by the pupil.]

*Suggestions.*—1. Since $AD \parallel BC$, compare $\angle 1$ and $\angle 2$.
2. Compare $\angle 3$ and $\angle 4$. What are the parallels?

**134. Cor. 1.** *The opposite sides of a parallelogram are equal.*

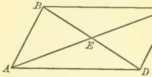
**135. Cor. 2.** *The opposite angles of a parallelogram are equal.*

**136. Cor. 3.** *Segments of parallels included between parallels are equal.*



## Proposition XXIV. Theorem

**137.** *The diagonals of a parallelogram bisect each other.*




**Hypothesis.** $ABCD$ is a $\square$.

Diagonals $AC$ and $BD$ intersect at $E$.

**Conclusion.** $AE = EC$; $BE = ED$.

[Proof to be given by the pupil.]

Note. — The point of intersection of the diagonals of a parallelogram is the **Center** of the parallelogram.

**Ex. 136.** If one angle of a parallelogram is 100°, how large is each of the other angles?

**Ex. 137.** If one angle of a parallelogram is a right angle, the others are also.

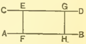
**Ex. 138.** If two adjacent sides of a parallelogram are equal, all its sides are equal.

**Ex. 139.** Two parallels are everywhere equidistant.

**Hypothesis.** $CD \parallel AB$.

$EF$ and $GH$ are any two ⊥s to $CD$ and $AB$.

**Conclusion.** $EF = GH$.




**Ex. 140.** If perpendiculars $BE$ and $DF$ are drawn to the diagonal $AC$ of a parallelogram $ABCD$, then $BE = DF$.

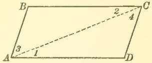
(Construct the figure with ruler and compasses.)

**Ex. 141.** If a line be drawn through the center of a parallelogram and terminated by two opposite sides of the parallelogram, it is bisected by the center.

**Ex. 142.** Construct the parallelogram whose diagonals are 2 in. and 3 in. respectively if the included acute angle is 45°. Measure the longer and shorter sides of the parallelogram.

## PROPOSITION XXV. THEOREM

**138**. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*




**Hypothesis.** $AB = CD$; $AB \parallel CD$.

**Conclusion.** $ABCD$ is a parallelogram.

**Plan.** $AD$ must be proved $\parallel$ to $BC$. Try to prove $\angle 1 = \angle 2$.

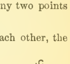
| | | |
|---|---|---|
| **Proof.** 1. | In $\triangle ABC$ and $\triangle ACD$: | |
| | $AB = CD$ and $AC = AC$; | Why? |
| | $\angle 3 = \angle 4$. | |
| | [Since $\parallel$s $AB$ and $CD$ are cut by $AC$.] | Why? |
| 2. | $\therefore \triangle ABC \cong \triangle ACD$. | Why? |
| 3. | $\therefore \angle 1 = \angle 2$. | Why? |
| 4. | $\therefore AD \parallel BC$. | Why? |
| 5. | $\therefore ABCD$ is a parallelogram. | § 131 |


**Ex. 143.** The line joining the mid-points of two opposite sides of a parallelogram is parallel to the other two sides.
(Prove $AEFD$ is a $\square$ and therefore $EF \parallel AD$)

**Ex. 144.** If $ABCD$ is a parallelogram, and $E$ and $F$ are the mid-points of $AB$ and $CD$ respectively, then $AECF$ is also a parallelogram.

**Ex. 145.** Prove that two straight lines are parallel if any two points of one are equidistant from the other. (Recall § 84.)

**Ex. 146.** If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

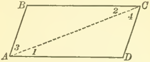



**Ex. 147.** If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, the perimeter of the parallelogram formed is equal to the sum of the equal sides of the triangle.

**Note.** — Supplementary Exercises 50–52, p. 277, can be studied now.

## Proposition XXVI. Theorem

**139.** *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*




**Hypothesis.** $AB = CD$; $BC = AD$.

**Conclusion.** $ABCD$ is a parallelogram.

**Plan.** Try to prove $AB \parallel CD$, and $AD \parallel BC$.

**Proof.** 1. $\triangle ABC \cong \triangle ACD$. Give the full proof.

2. $\therefore \angle 1 = \angle 2$, and hence $BC \parallel AD$. Why?

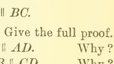
3. Also $\angle 3 = \angle 4$, and hence $AB \parallel CD$. Why?

4. $\therefore ABCD$ is a $\square$. Why?

**Note.** — Another proof may be given, which is based upon § 138.

**Ex. 148.** If $E$, $F$, $G$, and $H$ are mid-points of sides $AB$, $BC$, $CD$, and $AD$ respectively of parallelogram $ABCD$, then $EFGH$ is a parallelogram. (Prove $EF = HG$ and $EH = FG$.)

**Ex. 149.** Construct a parallelogram having sides 2 in. and 3 in. respectively, and with included angle of 45°. Measure its longer diagonal.




**140.** There is an important application of parallelograms in science. If an object is being pulled in the direction $AB$ with a force of 50 lb. and in the direction $AC$ with a force of 100 lb., it will actually move in the direction $AD$ and as if pulled by a force which bears to 100 lb. the same relation that $AD$ bears to $AC$.

Thus, $AC = 1''$ and $AD = 1\frac{1}{4}''$; since $AC$ represents 100 lb., $AD$ represents 125 lb.

**Ex. 150.** A steamer is being propelled east at the rate of 15 mi. an hour; the wind is driving it north at the rate of 5 mi. an hour. Determine by a construction the direction in which the steamer will travel and its rate.

## SPECIAL PARALLELOGRAMS

**141.** A **Rectangle** is a parallelogram one of whose angles is a right angle. It can be proved and it is important to remember that *all the angles of a rectangle are right angles.*

**Note.** — Since a rectangle is a special parallelogram, every theorem true about parallelograms is true about rectangles. Thus, the diagonals of a rectangle bisect each other. On the other hand, theorems true about a rectangle are not necessarily true about a parallelogram, since a rectangle is a *special* parallelogram.

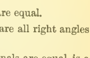
**Ex. 151.** State some other properties of a rectangle which follow at once from properties of a parallelogram. (See §§ 133–137.)

**Ex. 152.** Construct a rectangle whose sides are 1.5 in. and 2 in. respectively. Draw and measure its diagonals.

**Ex. 153.** Prove that the diagonals of a rectangle are equal.

**Ex. 154.** Prove that a quadrilateral whose angles are all right angles is a rectangle.

**Ex. 155.** Prove that a parallelogram whose diagonals are equal is a rectangle.




**Plan.** Try to prove one of its $\angle s$ is a right angle.

(Recall § 116, $G$ 1, and also § 103.)

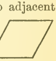
**Ex. 156.** When laying out the lines for the foundation of a rectangular building, as $ABCD$, contractors often measure off $AD$ and $DC$ at right angles and of the required lengths. Then $AB$ is measured off equal to $CD$ and at right angles to $AD$. (See figure for Ex. 155.)

($a$) Why should $ABCD$ then be a rectangle?

($b$) To test whether $ABCD$ is a true rectangle, $AC$ and $BD$ are measured. If they prove to be equal, it is concluded that the figure is a rectangle. Is this a safe test? Why?

**Note.** — Supplementary Exercises 53–54, p. 278, can be studied now.

**142.** A **Rhombus** is a parallelogram having two adjacent sides equal. It *can be proved* and it is important to remember that *all the sides of a rhombus are equal;* also it is *usually* implied that the angles are not right angles. (See § 143.)



**Ex. 157.** State properties of a rhombus which are evident at once because the rhombus is a special parallelogram. (See Note, § 141.)

**Ex. 158.** Prove that the diagonals of a rhombus are perpendicular to each other.

**Ex. 159.** Prove that the diagonals of a rhombus bisect the angles.

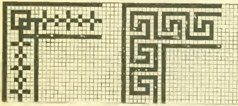
**Note.** — Supplementary Exercises 55–57, p. 278, can be studied now.

**143.** A **Square** is a parallelogram having two adjacent sides equal and one angle a right angle. It *can be proved* and it is important to remember that

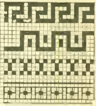
*All the angles of a square are right angles and all the sides are equal.*

**Note.** — The square is a special rectangle and also a special rhombus. Hence every theorem true about a rectangle or a rhombus is true about a square. (See Note, § 141.)

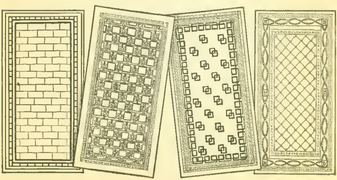
**144.** Many artistic designs are made on a network of squares as illustrated below.



Corner and Border



Four Border Designs



Rug Designs

**Ex. 160.** Make a list of facts about the square which may be inferred from known facts about the parallelogram, the rectangle, and the rhombus.

**Ex. 161.** How large are the angles into which a diagonal of a square divides its angles?

**Ex. 162.** Construct a square whose diagonals shall be 2 in. in length.

**Ex. 163.** Prove that the lines drawn from the ends of one side of a square to the mid-points of the two adjacent sides are equal.

**Ex. 164.** Prove that if the diagonals of a quadrilateral are perpendicnlar to and bisect each other, the figure is a rhombus.

**Ex. 165.** If $E$, $F$, $G$, and $H$ are points on the sides, $AB$, $BC$, $CD$, and $AD$ respectively of square $ABCD$, such that $AE = BF = CG = DH$, prove that $EFGH$ is a square.


*Suggestions.* — 1. Try to prove $EFGH$ is a $\square$, having two adj. sides equal, and having one $\angle$ ($\angle 4$) a right angle. (§ 143.)

2. To prove $\angle 4$ a right angle:

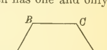
(a) $\angle 1 + \angle 2 =$ ? (b) Does $\angle 3 = \angle 2$?

(c) $\angle 1 + \angle 3 + \angle 4 =$ ? (d) $\therefore \angle 4 =$ ?

**Note.** — Supplementary Exercises 58–60, p. 278, can be studied now.

## TRAPEZOIDS

**145.** A **Trapezoid** is a quadrilateral which has one and only one pair of parallel sides; $AB$ and $CD$ are called the non-parallel sides.




The parallel sides of a trapezoid are called the **Bases.**

The perpendicular distance between the bases is called the **Altitude.**

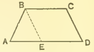
The line joining the mid-points of the non-parallel sides is called the **Median** of the trapezoid.

**146.** An **Isosceles Trapezoid** is a trapezoid the non-parallel sides of which are equal.

**Ex. 166.** Construct the trapezoid having lower base of 4 in., one of its non-parallel sides 2 in., the angle between these two sides being 60°, and the upper base being 1.5 in.

**Ex. 167.** If the angles at the ends of one base of a trapezoid are equal, the angles at the ends of the other base are also equal.

**Ex. 168.** If a trapezoid is isosceles, the lower base angles are equal. (If $AB = CD$, prove $\angle A = \angle D$. Draw $BE \parallel CD$. Compare $\angle AEB$ with $\angle D$ and $\angle A$.)




**Ex. 169.** If one pair of base angles of a trapezoid are equal, the trapezoid is isosceles.

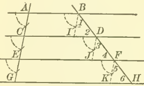
**Ex. 170.** Prove that the diagonals of an isosceles trapezoid are equal.

**Ex. 171.** Prove that the opposite angles of an isosceles trapezoid are supplementary.

**Note.** — Supplementary Exercises 61–63, p. 278, can be studied now.

## Proposition XXVII. Theorem

**147.** *If three or more parallels intercept equal lengths on one transversal, they intercept equal lengths on all transversals.*




**Hypothesis.** $AB \parallel CD \parallel EF \parallel GH$.
$AG$ cuts the ‖s at $A$, $C$, $E$, and $G$.
$BH$ cuts the ‖s at $B$, $D$, $F$, and $H$.
$AC = CE = EG$.

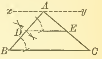
**Conclusion.** $BD = DF = FH$.

**Plan.** Try to prove $BD$, $DF$, and $FH$ homologous sides of cong. $\triangle$s.

| | | |
|---|---|---|
| **Proof.** 1. | Draw $BI$, $DJ$, $FK$ parallel to $AG$. | |
| 2. | $\therefore BI \parallel DJ \parallel FK$ | § 91 |
| 3. Also | $BI = AC$, $DJ = CE$, and $FK = EG$. | Why? |
| 4. But | $AC = CE = EG$. | Hyp. |
| 5. | $\therefore BI = DJ = FK$. | Ax. 1, § 51 |

[Complete the proof by proving $\triangle$s $BDI$, $DJF$, and $FHK$ are congruent, and then proving that $BD = DF = FH$.]

**148. Cor. 1.** *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.*

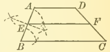



**Hyp.** $D$ is on $AB$ of $\triangle ABC$;
$AD = DB$; $DE \parallel BC$.

**Con.** $AE = EC$.

**Proof.** 1. Assume $XAY \parallel BC$.

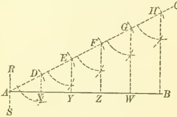
**149. Cor. 2.** *If a line is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, it bisects the other also.*




**Note.** — Supplementary Exercises 64–65, p. 278, can be studied now.

## Proposition XXVIII. Problem

**150.** *Divide a given segment into any number of equal parts.*




**Given** segment $AB$.

**Required** to divide $AB$ into five equal parts.

**Construction.** 1. Draw line $AC$, making a convenient $\angle$ with $AB$

2. Upon $AC$, lay off $AD = DE = EF = FG = GH$.

3. Draw $HB$.

4. Through $D$, $E$, $F$, $G$, and $H$, draw lines parallel to $HB$, meeting $AB$ at $X$, $Y$, $Z$, and $W$.

**Statement.** $AX = XY = YZ = ZW = WB$.

**Proof.** 1. Assume $RS$ through $A$ parallel to $HB$.

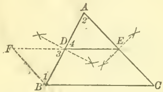
2. $\therefore RS \parallel DX \parallel EY \parallel FZ \parallel GW \parallel HB$. Why?

3. $\therefore AX = XY = YZ = ZW = WB$. Why?

**Note.** — Supplementary Exercises 66–67, p. 279, can be studied now.

## Proposition XXIX. Theorem

**151.** *If a line joins the mid-points of two sides of a triangle, it is parallel to the third side and equal to one half of it.*




**Hypothesis.** $D$ is the mid-point of $AB$, and $E$ is the mid-point of $AC$ in $\triangle ABC$.

**Conclusion.** $DE \parallel BC$; $DE = \frac{1}{2} BC$.

**Plan.** Extend $DE$ its own length to $F$. Try to prove $FE = BC$, and $FE \parallel BC$. To do this, try to prove $FECB$ is a $\square$.

| | | |
|---|---|---|
| **Proof.** 1. | Extend $DE$ to $F$, making $DF = DE$. Draw $BF$. | |
| 2. | $\therefore \triangle FBD \cong \triangle DAE$. | Give the proof. |
| 3. | $\therefore \angle 1 = \angle 2$; and also $BF = AE$. | Why? |
| 4. | $\therefore BF \parallel AC$, and $\therefore BF \parallel EC$. | Why? |
| 5. | Also $BF = EC$. | Why? |
| 6. | $\therefore BFEC$ is a parallelogram. | Why? |
| 7. | $\therefore FE$ or $DE$ is parallel to $BC$. | Why? |
| 8. | Also $FE = BC$, and $\therefore DE = \frac{1}{2} BC$. | Why? |

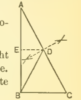
**Note.** — This theorem is *very* important.

**152.** The proof of Proposition XXIX illustrates another valuable device for proving theorems.

**Principle III.** To prove that one segment is double another, either double the shorter and prove the result equal to the longer, or halve the longer and prove the result equal to the shorter. The first of these plans is followed in the proof of Proposition XXIX; the second plan will be used in Proposition XL.

**Ex. 172.** The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

**Ex. 173.** If $E$, $F$, $G$, and $H$ are the mid-points of the sides $AB$, $BC$, $CD$, and $AD$ respectively of a quadrilateral $ABCD$, then $EFGH$ is a parallelogram. (Draw $AC$ and use Proposition XXIX.) This theorem appeared in a book on geometry by Th. Simpson in 1760.




**Ex. 174.** The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other.

**Ex. 175.** The mid-point of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. (Let $AE = EB$. Prove $ED \perp AB$. Then complete the proof.)

## Proposition XXX. Theorem

**153.** *The median of a trapezoid is parallel to the bases and equal to one half their sum.*


**Hypothesis.** $ABCD$ is a trapezoid.
$E$ is the mid-point of $AD$ and $F$ of $BC$.

**Conclusion.** $EF \parallel AB$ and $DC$.
$EF = \frac{1}{2}(AB + DC)$.

**Proof.** 1. Extend $DC$ to $G$, making $CG = AB$. Draw $AC$, $BG$, and $AG$.

2. $\therefore ABGC$ is a $\square$. Why?
[Since $CG = AB$, and $CG \parallel AB$.]

3. $\therefore AG$ passes through $F$ and is bisected by it. § 137

4. $\therefore$ in $\triangle ADG$, $AE = ED$ and $AF = FG$.

5. $\therefore EF \parallel DG$; and $EF = \frac{1}{2} DG$. Why?

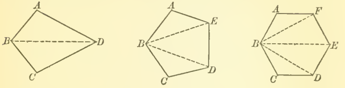
6. $\therefore EF \parallel DC$ and $AB$.

7. Also $EF = \frac{1}{2}(DC + AB)$, since $DG = DC + AB$.

**Note.** — Supplementary Exercises 68–74, p. 279, can be studied now.

## Proposition XXXI. Theorem

**154.** *The sum of the interior angles of a polygon having n sides is $(n-2)$ straight angles.*




**Hypothesis.** Assume a polygon of $n$ sides, like $ABCD \cdots$.

**Conclusion.** The sum of its int. $\measuredangle$s $= (n-2)$ st. $\measuredangle$s.

**Proof.** 1. Draw diagonals from $B$ to each of the other vertices.

2. Each side of the polygon, excepting $AB$ and $BC$, becomes the base of a triangle whose vertex is at $B$. Hence there are $(n-2)$ $\triangle$s formed.

That is, when $n$ is 4, there are 2 $\triangle$s;
when $n$ is 5, there are 3 $\triangle$s;
when $n$ is 6, there are 4 $\triangle$s; etc.

3. The sum of the int. $\measuredangle$s of each $\triangle$ is 1 st. $\angle$. Why?

4. $\therefore$ the sum of the int. $\measuredangle$s of the $(n-2)$ $\triangle$s is $(n-2)$ st. $\measuredangle$s.

5. But the sum of the int. $\measuredangle$s of the $\triangle$s = the sum of the int. $\measuredangle$s of the polygon.

6. $\therefore$ the sum of the int. $\measuredangle$s of the polygon is $(n-2)$ st. $\measuredangle$s.

**Ex. 176.** Express in straight angles, in right angles, and in degrees the sum of the angles of a polygon having:

(*a*) four sides; (*b*) five sides; (*c*) six sides; (*d*) eight sides.

**Ex. 177.** How many degrees are there in each angle of an equiangular:

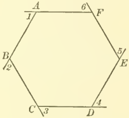
(*a*) quadrilateral? (*b*) pentagon? (*c*) hexagon? (*d*) octagon?

**Ex. 178.** If two angles of a quadrilateral are supplementary, then the other two are also.

**Ex. 179.** How many sides has a polygon the sum of whose angles is 16 right angles? 7 straight angles? 1620 degrees?

## Proposition XXXII. Theorem

**155.** *If the sides of any polygon be extended in order to form an exterior angle at each vertex, the sum of these exterior angles is two straight angles.*




**Hypothesis.** Assume a polygon of $n$ sides.
Extend the sides as in the figure.

**Conclusion.** The sum of ext. $\angle$s like $\angle 1$, $\angle 2$, $\angle 3$, etc. = 2 st. $\angle$s.

**Proof.** 1. The sum of the int. $\angle$ and the ext. $\angle$ at each vertex = 1 st. $\angle$. § 39

2. $\therefore$ the sum of all the int. and ext. $\angle$s $= n$ st. $\angle$s. Why?

3. But the sum of all the int. $\angle$s $=(n-2)$ st. $\angle$s. § 154

4. $\therefore$ the sum of all the ext. $\angle$s $= 2$ st. $\angle$s.

**Note.** — Propositions XXXI and XXXII were proved in their general form by Regiomontanus (1436–1476), although the theorems were known to earlier mathematicians and were proved by them for special cases.

**Ex. 180.** Prove the theorem of § 154 by drawing lines from any point within the polygon to the vertices. (Recall § 35.)

**Ex. 181.** State and prove the converse of § 135.
*Suggestion.* — Apply § 154 and § 98.

**Ex. 182.** How many sides are there in the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 540°?

**Ex. 183.** How many sides has a polygon the sum of whose interior angles equals four times the sum of its exterior angles?

### INEQUALITIES

**156.** The symbol for "less than" is $<$; for "greater than" is $>$.

**157. Order of Inequalities.** $a < b$ and $c < d$ are two inequalities of the same order. $m < n$ and $x > y$ are two inequalities of opposite orders.

**158. Axioms for combining Inequalities.**

Ax. 17. *If equals be added to unequals, the sums are unequal in the same order.*

Thus, if $a < b$, then $a + c < b + c$.

Ax. 18. *If equals be subtracted from unequals, the differences are unequal in the same order.*

Thus, if $a < b$, then $a - c < b - c$.

Ax. 19. *If unequals be added to unequals in the same order, the sums are unequal in the same order.*

Thus, if $a < b$, and $c < d$, then $a + c < b + d$.

Ax. 20. *If unequals be subtracted from equals or from unequals of opposite order, the differences are unequal and of order opposite to that of the subtrahend.*

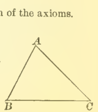
Thus, if $a > b$, and $c < d$, then $a - c > b - d$.

*Arithmetical Example.* — Since $12 > 7$ and $3 < 5$, then $12 - 3$ should be greater than $7 - 5$. Is it?

Ax. 21. *If $a > b$ and $b > c$, then $a > c$.*

**Ex. 184.** Given an arithmetical example for each of the axioms.

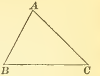
**159. Fundamental Inequalities for Segments.**




(*a*) *Any side of a triangle is less than the sum of the other two sides.*

This follows from Ax. 11, § 51.
Thus $BC < AB + AC$.

(*b*) *Any side of a triangle is greater than the difference between the other two sides.*




Thus, $BC > AC - AB$.

For, from (*a*) $BC + AB > AC$. Subtracting $AB$ from both members of the inequality, $BC > AC - AB$, by Ax. 18, § 158.

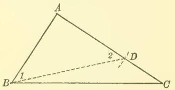
Note. — Ex. 188, p. 86, and Supplementary Exercises 75–84, p. 280, can be studied now.

**160. Fundamental Inequality for Angles.**

*An exterior angle of a triangle is greater than either remote interior angle of the triangle.* (§ 87.)

## Proposition XXXIII. Theorem

**161.** *If two sides of a triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.*




| | | |
|---|---|---|
| **Hypothesis.** | In $\triangle ABC$, $AC > AB$. | |
| **Conclusion.** | $\angle B > \angle C$. | |

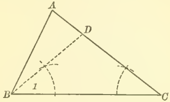
**Proof.** 1. Since $AC > AB$, take $AD = AB$. Draw $BD$. § 13

| | | |
|---|---|---|
| 2. | $\therefore \angle 1 = \angle 2$. | Why? |
| 3. | $\angle 2$ is an exterior angle of $\triangle BDC$. | Def. |
| 4. | $\therefore \angle 2 > \angle C$. | Why? |
| 5. | $\therefore \angle 1 > \angle C$. | Why? |
| 6. But | $\angle ABC > \angle 1$. | Ax. 8, § 51 |
| 7. | $\therefore \angle ABC > \angle C$. | Ax. 21, § 158 |

**Ex. 185.** If a triangle is scalene, all its angles are unequal.

## Proposition XXXIV. Theorem

**162.** *If two angles of a triangle are unequal, the sides opposite are unequal, the side opposite the greater angle being the greater.*




**Hypothesis.** In $\triangle ABC$, $\angle C < \angle B$.

**Conclusion.** $AB < AC$.

**Proof.** 1. Since $\angle B > \angle C$, construct $BD$, making $\angle 1 = \angle C$.

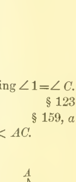
2. $\therefore BD = DC$. § 123

3. In $\triangle ABD$, $AB < AD + BD$. § 159, *a*

4. $\therefore AB < AD + DC$, or $AB < AC$.

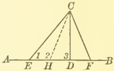
[Substitute $DC$ for its equal, $DB$.]

**163. Cor. 1.** *The hypotenuse of a right triangle is greater than either leg of the triangle.*




**164. Cor. 2.** *The perpendicular from a point to a line is the shortest segment from the point to the line.*

**165. Cor. 3.** *If two oblique segments, drawn from a point in a perpendicular to a line, cut off unequal distances from the foot of the perpendicular, the more remote is the greater.*




**Hyp.** $CD \perp AB$; $ED > DF$. **Con.** $CE > CF$.

*Suggestions.* — 1. Take $DH = DF$, and draw $CH$.

2. Prove $CH = CF$.

3. Prove $\angle 2 > \angle 1$, by comparing each with $\angle 3$.

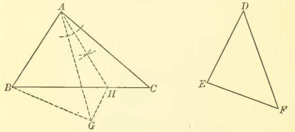
4. Then complete the proof.

**Ex. 186.** If $O$ is any point in the base $BC$ of isosceles triangle $ABC$, then $AO$ is less than $AC$. (Prove $\angle AOC > \angle ACO$.)

**Ex. 187.** Prove that the median to any side of a triangle is greater than the altitude to that side unless the side is the base of an isosceles triangle.

## Proposition XXXV. Theorem

**166.** *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*




**Hypothesis.** In $\triangle ABC$, and $\triangle DEF$:

$$AB = DE;\ AC = DF;\ \angle BAC > \angle D.$$

**Conclusion.** $BC > EF.$

**Proof.** 1. Place $\triangle DEF$ in the position $ABG$, side $DE$ coinciding with its equal $AB$.

2. $DF$ falls within $\angle BAC$, taking the position $AG$. Why?

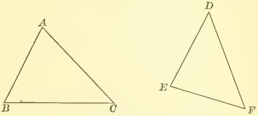
3. Construct $AH$ bisecting $\angle GAC$, and meeting $BC$ at $H$. Draw $GH$.

| | | |
|---|---|---|
| 4. | $\triangle GAH \cong \triangle ACH.$ | Give the full proof. |
| 5. | $\therefore GH = CH.$ | Why? |
| 6. | In $\triangle BHG$, $BH + GH > BG.$ | Why? |
| 7. | $\therefore BH + CH > BG$, or $BC > BG.$ | Why? |

**Note.** — If $G$ falls on $BC$, then $EF$ is at once less than $BC$. If $G$ falls within $\triangle ABC$, the proof is similar to that given in the text.

## Proposition XXXVI. Theorem

**167.** *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*




**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:

$AB = DE$; $AC = DF$; $BC > EF$.

**Conclusion.** $\angle A > \angle D$.

**Proof.** 1. Suppose that $\angle A$ is not greater than $\angle D$; that is, that $\angle A$ either equals $\angle D$ or is less than $\angle D$.

| | | |
|---|---|---|
| 2. | If $\angle A = \angle D$, then $\triangle ABC \cong \triangle DEF$. [Give the full proof.] | |
| 3. | Then $BC = EF$. | Why? |
| 4. | But $BC > EF$. | Hyp. |
| 5. | $\therefore \angle A$ cannot be equal to $\angle D$. | § 94 |
| 6. | If $\angle A < \angle D$, then $BC < EF$. | § 166 |
| 7. | But $BC > EF$. | Hyp. |
| 8. | $\therefore \angle A$ cannot be less than $\angle D$. | |

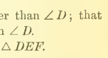
9. Since $\angle A$ cannot be equal to $\angle D$ or less than $\angle D$, then $\angle A$ must be greater than $\angle D$.

**Ex. 188.** If $O$ is any point within $\triangle ABC$, then $BO + OC < BA + AC$.

*Suggestions.* — 1. Extend $CO$ until it intersects $AB$ at $R$. 2. Compare $BO$ with $BR + RO$.

3. Add $OC$ to hoth memhers of the inequality.

4. Compare $RO + OC$ with $RA + AC$, and complete the proof.

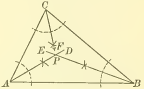


### SUPPLEMENTARY THEOREMS

**168.** Three or more lines ordinarily do not pass through a common point. Three or more lines which *do pass* through a common point are called **Concurrent Lines.**

## PROPOSITION XXXVII. THEOREM

**169.** *The bisectors of the interior angles of a triangle meet at a point which is equidistant from the sides of the triangle.*




**Hypothesis.** In $\triangle ABC$:

$AD$ bisects $\angle A$; $BE$ bisects $\angle B$; $CF$ bisects $\angle C$.

**Conclusion.** $AD$, $BE$, and $CF$ meet at a point which is equidistant from the sides of $\triangle ABC$.

**Proof.** 1. Let $AD$ and $BE$ meet at point $P$. Note 1

2. Since $P$ is in $AD$, it is equidistant from $AC$ and $AB$. § 120, I

3. Since $P$ is in $BE$, it is equidistant from $AB$ and $BC$.

4. $\therefore P$ is equidistant from $AC$ and $BC$. Ax. 1, § 51

5. $\therefore P$ lies in $CF$, the bisector of $\angle C$. § 120, II

6. Hence $AD$, $BE$, and $CF$ meet at $P$, which is equidistant from $AB$, $AC$, and $BC$.

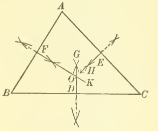
**Note 1.** — This fact may be assumed as evident from the figure, or may be proved as follows:

1. If $AD$ does not intersect $BE$, then $AD \parallel BE$.
2. Then $\angle DAB + \angle EBA = 1$ st. $\angle$. (§ 103.)
3. But this is impossible, since $\angle DAB + \angle EBA < 1$ st. $\angle$. Why?

**Note 2.** — The point of intersection of the bisectors of the interior angles of a triangle is called the **In-center** of the triangle. (See § 226.)

## Proposition XXXVIII. Theorem

**170.** *The perpendicular-bisectors of the sides of a triangle meet at a point which is equidistant from the vertices of the triangle.*




**Hypothesis.** In $\triangle ABC$, $FK$, $DG$, and $EH$ are the perpendicular-bisectors of $AB$, $BC$, and $AC$, respectively.

**Conclusion.** $FK$, $DG$, and $EH$ meet at a point which is equidistant from $A$, $B$, and $C$.

**Proof.** 1. Let $FK$ and $DG$ meet at point $O$. Note 1

2. Since $O$ is in $FK$, $O$ is equidistant from $A$ and $B$. Why?

3. Since $O$ is in $DG$, $O$ is equidistant from $B$ and $C$. Why?

4. $\therefore O$ is equidistant from $A$ and $C$. Ax. 1, § 51

5. $\therefore O$ lies in $EH$, or $EH$ passes through $O$. § 118

6. Hence the perpendicular-bisectors are concurrent at a point which is equidistant from $A$, $B$, and $C$.

**Note 1.** — This fact may be assumed or be proved as follows:

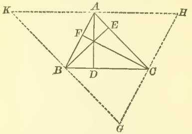
1. If $FK$ does not intersect $GD$, then $FK \parallel GD$.
2. $\therefore AB$, which is $\perp$ to $FK$, is also $\perp$ to $GD$.
3. But $BD \perp GD$.
4. $\therefore$ either $AB \parallel BD$, or $AB$ coincides with $BD$.
5. But this is impossible, since $AB$ and $BD$ intersect.

**Note 2.** — The point of intersection of the perpendicular-bisectors of the sides of a triangle is called the **Circum-center** of the triangle, for a circle can be drawn with it as center which will pass through the vertices of the triangle.

**Ex. 189.** Construct a circle which will pass through the vertices of the triangle the sides of which are 3 in., 3 in., and 4 in., respectively.

## Proposition XXXIX. Theorem

**171.** *The altitudes of a triangle meet at a point.*




**Hypothesis.** In $\triangle ABC$, $AD$, $BE$, and $CF$ are the altitudes from $A$, $B$, and $C$ respectively.

**Conclusion.** $AD$, $BE$, and $CF$ meet at a point.

**Proof.** 1. Draw $HK$ through $A \parallel$ to $BC$; $KG$ through $B \parallel$ to $AC$; and $GH$ through $C \parallel AB$. These parallels form $\triangle HKG$.

2. Since $AD \perp BC$, then $AD \perp HK$. Why?

3. $KACB$ and $ABCH$ are ▱. Why?

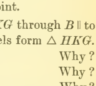
4. $KA = BC$, and $AH = BC$. Why?

5. $\therefore KA = AH$, and $AD$ is the perpendicular-bisector of $KH$.

6. Similarly $BE$ and $CF$ can be proved to be the perpendicular-bisectors of $KG$ and $GH$ respectively.

7. $\therefore AD$, $BE$, and $CF$ in $\triangle HKG$ meet at a point. § 170

**Note.** — The point of intersection of the altitudes of a triangle is called the **Ortho-center** of the triangle.




**Ex. 190.** If $I$ is the ortho-center (Note, § 171) and $J$ is the circum-center (Note, § 170) of triangle $ABC$, then $BI = 2\ JK$ and $AI = 2\ JL$.

*Suggestions.* — 1. Recall § 152.

2. Prove $MN \parallel KL$, $BI \parallel JK$, and $AI \parallel JL$.

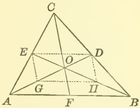
3. Recall § 105.

**Ex. 191.** Does the ortho-center of a triangle necessarily fall inside the triangle?

**Note.** — Supplementary Exercise 85, p. 281, can be studied now.

## Proposition XL. Theorem

**172.** *The medians of a triangle meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.*




**Hypothesis.** $AD$, $BE$, and $CF$ are the medians of $\triangle ABC$.

**Conclusion.** $AD$, $BE$, and $CF$ meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.

**Proof.** 1. Let $AD$ and $BE$ meet at point $O$. Note 1, § 169.

2. Let $G$ and $H$ be the mid-points of $AO$ and $BO$ respectively. Draw $ED$, $GH$, $EG$, and $DH$.

3. Then, in $\triangle AOB$, $GH = \frac{1}{2} AB$ and $GH \parallel AB$. Why?

4. Similarly, $ED = \frac{1}{2} AB$ and $ED \parallel AB$.

5. $\therefore EDHG$ is a $\square$. Why?

6. $\therefore GD$ and $EH$ bisect each other. Why?

7. $\therefore OD = OG = AG$, and $EO = OH = HB$.

8. Hence $AD$ and $BE$ meet at a point which lies two thirds the distance from $A$ to $D$ and from $B$ to $E$.

9. In like manner, $AD$ and $CF$ meet at a point which lies two thirds the distance from $A$ to $D$ and from $C$ to $F$. On $AD$, this is point $O$.

10. Hence the three medians meet at point $O$, which is two thirds the distance from each vertex to the mid-point of the opposite side.

**Note.** — The point of intersection of the medians of a triangle is called the **Center of Gravity** of the triangle.

This theorem was known to Archimedes.

### Exercises Solved by Indirect Proofs

**Ex. 192.** If two straight lines are cut by a transversal, and a pair of alternate interior angles are unequal, the lines are not parallel.

*Suggestion.* — Review § 94.

**Ex. 193.** If two lines are cut by a transversal and the sum of the interior angles on the same side of the transversal is not equal to two right angles, the lines are not parallel.

**Ex. 194.** If a point is unequally distant from the ends of a segment, it is not in the perpendicular-bisector of the segment.

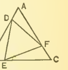
**Ex. 195.** If a point is not equidistant from the sides of an angle, it is not in the bisector of the angle.

**Ex. 196.** Prove that the two altitudes of a parallelogram which has two unequal sides are unequal.

**Ex. 197.** If two unequal oblique segments be drawn from a point to a straight line, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.

*Suggestion.* — Recall § 165.

### Miscellaneous Exercises

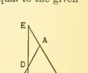



**Ex. 198.** If $D$, $E$, and $F$ are points on the sides $AB$, $BC$, and $AC$ respectively of equilateral triangle $ABC$, such that $AD = BE = CF$, then $\triangle DEF$ is also equilateral.

**Ex. 199.** Prove that the bisectors of a pair of vertical angles form a straight line.

**Ex. 200.** If two lines are cut by a transversal so that a pair of exterior angles on the same side of the transversal are supplementary, the lines are parallel.

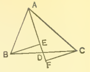
**Ex. 201.** If perpendiculars be drawn to the sides of an acute angle from a point outside of the angle, they form an angle equal to the given angle.




**Ex. 202.** If through any point $D$ in one of the equal sides $AB$ of isosceles $\triangle ABC$, $DF$ be drawn perpendicular to base $BC$, meeting $CA$ extended at $E$, then $\triangle ADE$ is isosceles.

*Suggestion.* — Compare $\angle E$ with $\angle C$, and $\angle BDF$ with $\angle B$.

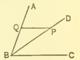
**Ex. 203.** Prove that the altitudes drawn to homologous sides of congruent triangles are equal.




**Ex. 204.** If $D$ is mid-point of side $BC$ of $\triangle ABC$, and $BE$ and $CF$ are perpendiculars from $B$ and $C$ to $AD$, extended if necessary, prove $BE = CF$.

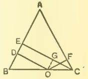
**Ex. 205.** If a line be drawn through the vertex of an isosceles triangle parallel to the base, it bisects the exterior angle at the vertex.

**Ex. 206.** Prove that the segments bisecting the base angles of an isosceles triangle and terminating in the opposite sides are equal.




**Ex. 207.** If a line be drawn through a point in the bisector of an angle parallel to one side of the angle, the bisector, the parallel, and the other side of the angle form an isosceles triangle.

**Ex. 208.** If the median to the base of a triangle is perpendicular to the base, the triangle is isosceles.

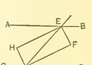



**Ex. 209.** Prove that the sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides of the triangle is equal to the altitude drawn to one of the equal sides.

Prove $OD + OF = CE$.

*Suggestions.* — 1. Draw $OG \perp CE$.

2. Compare $OD$ and $EG$; also $OF$ and $CG$.




**Ex. 210.** If two parallels are cut by a transversal, the bisectors of the four interior angles form a rectangle.

*Suggestions.* — 1. $EFGH$ must be proved a $\square$ and one $\angle$ must be proved a right angle.

2. Recall §§ 93, 103, 106.

**Ex. 211.** If the mid-point of any side of a square is joined to the two vertices of the opposite side, the lines so drawn are equal.

**Ex. 212.** Prove that the lines drawn from the mid-point of the base of an isosceles triangle to mid-points of the sides of the triangle form with the half sides a rhombus.

**Ex. 213.** If the lower base $AD$ of trapezoid $ABCD$ is double the upper base $BC$, and the diagonals intersect at $E$, prove that $CE$ is $\frac{1}{2}$ $AE$ and that $BE$ is $\frac{1}{2}$ $ED$. (§ 152.)

**Ex. 214.** If $D$ is any point in side $AC$ of $\triangle ABC$ and $E$, $F$, $G$, and $H$ are the mid-points of $AD$, $CD$, $BC$, and $AB$, respectively, then $EFGH$ is a parallelogram.

*Suggestion.* — Draw $BD$.

**Note.** — Supplementary Exercises 86–108, p. 281, can be studied now.

# BOOK II

## THE CIRCLE

**173.** Review the definitions given in § 16 and § 17, and the Exercises 31–34, Introduction. The symbol for circle is $\odot$. The circle whose center is $O$, is denoted by $\odot\ O$.

**174.** Since a circle is a *closed line* (§ 6), it incloses a portion of the plane called its interior.

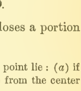
**Ex. 1.** Draw a circle with radius 1 in. Where will a point lie: (*a*) if its distance from the center is $\frac{3}{4}$ in.? (*b*) If its distance from the center is 1.5 in.?

**Ex. 2.** Draw a circle with diameter 5 in. Cut the circle from paper. Prove, by folding it, that any diameter bisects the circle and the surface within the circle.

**Ex. 3.** Draw two circles which intersect. From one of the points of intersection draw the radius of each circle. How does the distance between their centers compare with the sum of their radii?

**Ex. 4.** Prove that a diameter of a circle is greater than any other chord of the circle.

*Suggestion.* — Compare $CD$ with $AO$ and $OB$. Also, compare $AO + OB$ with $AB$.




**Ex. 5.** If two circles intersect, the distance between their centers is greater than the difference of their radii.

**175.** From the exercises following § 17 and § 174, the following facts are evident:

(*a*) *If a straight line cuts a circle, it intersects it in two and only two points.*

(*b*) *If two circles intersect, they have two and only two points of intersection.*

(c) *A point is within, on, or outside a circle if its distance from the center is less than, equal to, or greater than the radius.*

(d) *A diameter of a circle bisects the circle and the surface within it; also, if a line bisects a circle, it is a diameter of the circle.*

The theorem (d) was known to Thales.

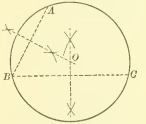
**176.** One half of a circle is called a **Semicircle**.

A quarter of a circle is called a **Quadrant**.

Circles having the same center are called **Concentric Circles**.

## Proposition I. Problem

**177.** *Construct a circle which will pass through three points which are not in a straight line.*




**Given** points $A$, $B$, and $C$ which are not in a straight line.

**Required** to construct a circle which will pass through $A$, $B$, and $C$.

**Construction.** 1. Draw $AB$ and $BC$.

2. Construct the $\perp$ bisectors of $AB$ and $BC$, meeting at $O$.

**Statement.** A circle drawn with $O$ as center and $OA$ as radius will pass through $A$, $B$, and $C$.

**Proof.** $OA = OB = OC$. Why?

Note. — Only one circle can be drawn through three points, for the center must lie on each of the perpendicular-bisectors (§ 118, II) and these lines can intersect at only one point.

**Ex. 6.** What would happen if the three points in Proposition I were in a straight line?

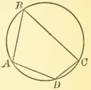
**Ex. 7.** (*a*) Construct a circle which will pass through two given points.

(*b*) How many circles can be constructed through two given points?

(*c*) Where do the centers of all these circles lie?

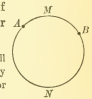
**Ex. 8.** Construct full size the design for a four-inch square tile. Make the decorative arcs $\frac{3}{8}$ in. wide.

**178.** A **polygon** is said to be **inscribed in a circle** when its vertices lie on the circle; as *ABCD*. The **circle** is said to be **circumscribed about the polygon.**




CHORDS, ARCS, AND CENTRAL ANGLES

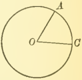
**179.** Two points on a circle are the ends of two arcs; a **Minor Arc**, as *AMB*, and a **Major Arc**, as *ANB*.




Unless the contrary is stated, the minor arc will always be understood when an arc is indicated by means of its extremities. Thus, arc *AB* means minor arc *AB*.

An arc *AB* will be indicated by a small arc drawn over *AB*; as $\overset{\frown}{AB}$.

**180.** A **Central Angle** is an angle whose vertex is at the center and whose sides are radii of the circle; as $\angle AOC$.




$\angle AOC$ is said to *intercept* $\overset{\frown}{AC}$.

$\overset{\frown}{AC}$ is said to be *intercepted by* $\angle AOC$.

"Intercept" is derived from two Latin words meaning "between" and "to take," so that it means "to take between."

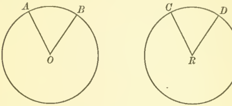
**Ex. 9.** Construct a circle with radius 2 in. (*a*) Construct two central angles which are equal, and a third central angle which is greater than each of the equal central angles. (*b*) Cut from the paper the two equal central angles. Compare their intercepted arcs by superposition.

(*c*) Cut from the paper the third central angle; compare its intercepted arc with the arc intercepted by one of the other two angles.

(*d*) What do you conclude must be true about the arcs intercepted by equal central angles of a circle? By unequal central angles?

## Proposition II. Theorem

**181.** *In the same circle or in equal circles, if central angles are equal, they intercept equal arcs.*




**Hypothesis.** $\odot O = \odot R$; $\angle AOB = \angle CRD$.

**Conclusion.** $\overset{\frown}{AB} = \overset{\frown}{CD}$.

**Proof.** 1. Place $\odot O$ on $\odot R$, with point $O$ on point $R$, and so that $\angle AOB$ coincides with its equal, $\angle CRD$.

2. Then $\odot O$ will coincide with $\odot R$. § 17

3. Point $A$ will fall on point $C$, since $OA = RC$. § 17

4. Point $B$ will fall on point $D$, since $OB = RD$. § 17

5. $\therefore \overset{\frown}{AB}$ coincides with $\overset{\frown}{CD}$ and hence $\overset{\frown}{AB} = \overset{\frown}{CD}$.

**Ex. 10.** Divide a circle into four equal arcs.
What kind of central angles must be constructed?

**Ex. 11.** Divide a circle into eight equal arcs.

**Ex. 12.** Divide a circle into six equal arcs.

How large must the central angles be? Recall § 70 and Ex. 113, Book I.

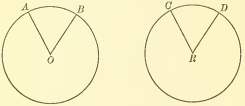
**Ex. 13.** Using the construction made in Ex. 12, draw a six-pointed star.

**Ex. 14.** Tell how you can divide a circle into five equal parts *by means of your protractor* and straightedge.

**Ex. 15.** By means of compass, ruler, and protractor, draw a five-pointed star in a circle with 2 in. radius, to be used as a pattern for a star on a sailor collar.

## Proposition III. Theorem

**182.** *In the same circle or in equal circles, if arcs are equal, the central angles which intercept them are equal.*




**Hypothesis.** $\odot O = \odot R$; $\widehat{AB} = \widehat{CD}$.

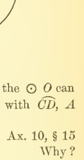
**Conclusion.** $\angle AOB = \angle CRD$.

**Proof.** 1. Since $\odot O = \odot R$, and $\widehat{AB} = \widehat{CD}$, the $\odot O$ can be made to coincide with the $\odot R$, and $\widehat{AB}$ with $\widehat{CD}$, $A$ falling on $C$, $B$ on $D$, and $O$ on $R$.

2. $\therefore AO$ will fall on $CR$ and $BO$ on $DR$. Ax. 10, § 15

3. $\therefore \angle AOB = \angle CRD$. Why?

**Ex. 16.** If a radius bisects an arc, it is perpendicular to and bisects the chord which subtends the arc.




**Ex. 17.** The twelve spokes of a wheel are spaced so that the points at which they are attached to the rim divide the rim into equal arcs. How many degrees are there in the angle formed by two adjacent spokes?

**183.** It may be proved that: *in the same circle or in equal circles,*

(*a*) *The greater of two unequal central angles intercepts the greater arc;*

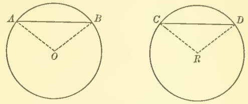
(*b*) *The greater of two unequal arcs is intercepted by the greater central angle.*

**184.** A **chord** $AB$ is said to **subtend** arc $AB$. Arc $AB$ is said **to be subtended** by **chord** $AB$.

"Subtend" is derived from Latin words meaning "to stretch under."

## Proposition IV. Theorem

**185.** *In the same circle or in equal circles, if chords are equal, they subtend equal arcs.*




**Hypothesis.** $\odot O = \odot R$; $AB = CD$.

**Conclusion.** $\widehat{AB} = \widehat{CD}$.

**Plan.** 1. Draw $AO$, $OB$, $RC$, $RD$.

2. Prove $\angle O = \angle R$, and apply § 181.

[Proof to be given by the pupil.]

**Ex. 18.** Construct an equilateral triangle. Circumscribe a circle about the triangle. (§ 177.) Prove that the vertices of the triangle divide the circle into three equal arcs.

## Proposition V. Theorem

**186.** *In the same circle or in equal circles, if arcs are equal, the chords which subtend them are equal.*

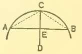
**Hypothesis.** $\odot O = \odot R$; $\widehat{AB} = \widehat{CD}$. (Fig. § 185.)

**Conclusion.** $AB = CD$.

**Plan.** Try to prove $\triangle AOB \cong \triangle CRD$. Compare $\angle O$ and $\angle R$. § 182

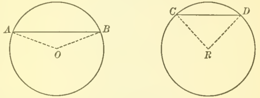
[Proof to be given by the pupil.]

**Ex. 19.** If $C$ is the mid-point of arc $AB$, prove that $AC$ is greater than one half $AB$.
Draw $CB$. Compare $AB$ with $AC + CB$.



## Proposition VI. Theorem

**187.** *In the same circle or in equal circles, if two minor arcs are unequal, then their chords are unequal, the greater arc being subtended by the greater chord.*




**Hypothesis.** $\odot O = \odot R$; $\widehat{AB} > \widehat{CD}$.

**Conclusion.** $AB > CD$.

**Proof.** 1. Draw radii $AO$, $BO$, $CR$, and $DR$.

2. In $\triangle AOB$ and $\triangle CRD$:

$AO = CR$ and $BO = DR$; Hyp.

but since $\widehat{AB} > \widehat{CD}$, $\angle O > \angle R$. § 183, *b*

3. $\therefore AB > CD$. § 166

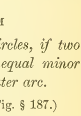
## Proposition VII. Theorem

**188.** *In the same circle or in equal circles, if two chords are unequal, then they subtend unequal minor arcs, the greater chord subtending the greater arc.*

**Hypothesis.** $\odot O = \odot R$; $AB > CD$. (Fig. § 187.)

**Conclusion.** $\widehat{AB} > \widehat{CD}$.

**Plan.** Prove $\angle O > \angle R$ (§ 167) and apply § 183, *a*.

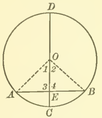



**Ex. 20.** Prove that the straight line which bisects the arcs subtended by a chord bisects the chord at right angles.

*Suggestion.*—Compare $AD$ and $BD$; also $AC$ and $BC$. Apply § 77.

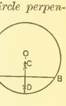
## Proposition VIII. Theorem

**189.** *If a diameter is perpendicular to a chord, it bisects the chord and its subtended arcs.*




| | | |
|---|---|---|
| **Hypothesis.** | In $\odot O$, diameter $CD \perp AB$ at $E$. | |
| **Conclusion.** | $AE = EB$; $\widehat{AC} = \widehat{CB}$; and $\widehat{AD} = \widehat{DB}$. | |
| **Proof.** 1. | Draw $AO$ and $OB$. | |
| 2. | $\triangle AEO \cong \triangle OEB$. | Give the full proof. |
| 3. | $\therefore AE = EB$, and $\angle 1 = \angle 2$. | Why? |
| 4. | $\therefore \widehat{AC} = \widehat{CB}$. | § 181 |
| 5. Also, | $\widehat{AD} = \widehat{DB}$. | Why? |

**190. Cor. 1.** *A line through the center of a circle perpendicular to a chord bisects the chord.*



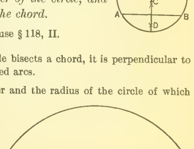

**191. Cor. 2.** *The perpendicular-bisector of a chord passes through the center of the circle, and bisects the arcs subtended by the chord.*

Compare $AO$ and $OB$. Then use § 118, II.

**Ex. 21.** If a radius of a circle bisects a chord, it is perpendicular to the chord and bisects the subtended arcs.

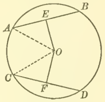
**Ex. 22.** Determine the center and the radius of the circle of which $\widehat{AB}$ is an arc.

Draw any two chords and erect the $\perp$ bisectors. These must pass through the center. (§ 191.)



## Proposition IX. Theorem

**192.** *In the same circle or in equal circles, if chords are equal, they are equidistant from the center.*




**Hypothesis.** In $\odot ABC$:

$AB = CD$; $OE \perp AB$; $OF \perp CD$.

**Conclusion.** $OE = OF$.

**Proof.** 1. Draw $OA$ and $OC$.

2. $AE = \frac{1}{2} AB$, $CF = \frac{1}{2} CD$, and $AB = CD$. Why?

3. $\therefore AE = CF$. Ax. 6, § 51

4. $\therefore \triangle AEO \cong \triangle CFO$. Give full proof.

5. $\therefore OE = OF$. Why?

**Ex. 23.** If two intersecting chords are equal, the radius drawn through the point of intersection bisects the angle between them.

**Ex. 24.** If a straight line bisects a chord and its subtended arc, then it is perpendicular to the chord. (§ 186.)

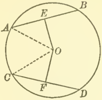
**Ex. 25.** If a straight line is drawn cutting two concentric circles in the points $A$, $B$, $C$, and $D$, respectively, then $AB$ equals $CD$.

**Ex. 26.** Prove that the perpendicular-bisectors of the sides of an inscribed polygon are concurrent. (§ 168.)

**Ex. 27.** On equal chords of a circle, points are taken at equal distances from the ends of the chords. Prove that all these points are equidistant from the center of the circle.

## Proposition X. Theorem

**193.** *In the same circle or in equal circles, if chords are equidistant from the center, they are equal.*




**Hypothesis.** In $\odot ABC$:

$OE \perp AB$; $OF \perp CD$; $OE = OF$.

**Conclusion.** $AB = CD$.

**Proof.** 1. Draw $OA$ and $OC$.

2. Then $\triangle OEA \cong \triangle OCF$ and $AE = CF$.

[Give the full proof.]

3. But $AB = 2\ AE$ and $CD = 2\ CF$. Why?

4. $\therefore AB = CD$. Why?

**Ex. 28.** If a straight line be drawn parallel to the line connecting the centers of two equal circles, and intersecting the circles, then the chords formed in the two circles are equal.

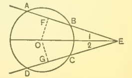
**Ex. 29.** If two chords, intersecting within the circle, make equal angles with the radius passing through their point of intersection, they are equal.

*Suggestion.* — Draw ⊥s to the chords from the center; prove the ⊥s are equal; then use § 193.

**Ex. 30.** If two equal chords of a circle intersect, the segments of one equal respectively the segments of the other.

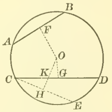
**194.** A straight line which intersects a circle in two points is called a **Secant** of the circle.

**Ex. 31.** If $ABE$ and $DCE$ are two secants of a circle which make equal angles with the line connecting $E$ with the center of the circle, then chord $AB$ = chord $DC$. (Use § 193.)



## Proposition XI. Theorem

**195.** *In the same circle or in equal circles, the less of two unequal chords is at the greater distance from the center of the circle.*




**Hypothesis.** In $\odot O$:

$AB < CD$; $OF \perp AB$; $OG \perp CD$.

**Conclusion.** $OF > OG$.

| | | |
|---|---|---|
| **Proof.** 1. | Since $AB < CD$, $\overset{\frown}{AB} < \overset{\frown}{CD}$. | § 188 |
| 2. | Let $\overset{\frown}{CE} = \overset{\frown}{AB}$. Draw $CE$. | |
| 3. | $\therefore CE = AB$. | Why? |
| 4. | Draw $OH \perp CE$, intersecting $CD$ at $K$. | |
| 5. | $\therefore OH = OF$. | § 192 |
| 6. | But $OH > OK$. | Why? |
| 7. | $\therefore OF > OK$. | Why? |
| 8. | $OK > OG$. | Why? |
| 9. | $\therefore OF > OG$. | Why? |

**Ex. 32.** An equilateral triangle and a square are inscribed in a circle. Prove that the sides of the triangle are nearer the center than the sides of the square.

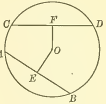
**Ex. 33.** Chord $BY$ is drawn through one extremity of a diameter $AB$ of circle $O$. Radius $OX$ is drawn in $\angle ABY$ parallel to $BY$, intersecting arc $AY$ at $X$. Prove arc $AX$ equals arc $XY$.

*Suggestion.* — Draw radius $OY$.

**Ex. 34.** $AB$ is a diameter of a circle and $XY$ is an intersecting diameter of a smaller concentric circle. Prove $AXBY$ is a parallelogram.

## Proposition XII. Theorem

**196.** *In the same circle or in equal circles, if two chords are unequally distant from the center, the more remote is the smaller.*




**Hypothesis.** In $\odot O$:

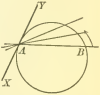
$OE \perp AB$; $OF \perp CD$; $OE > OF$.

**Conclusion.** $AB < CD$.

**Proof.** 1. Suppose that $AB$ is not less than $CD$; that is, suppose that $AB = CD$ or $AB > CD$.

| | | |
|---|---|---|
| 2. | If $AB = CD$, then $OE = OF$. | Why? |
| | But $OE > OF$. | Why? |
| | $\therefore AB$ is not $=$ to $CD$. | |
| 3. | If $AB > CD$, then $OE < OF$. | § 195 |
| | But $OE > OF$. | Hyp. |
| | $\therefore AB$ is not greater than $CD$. | |
| 4. | $\therefore AB < CD$. | |

**197. Tangent Line.** Assume that the secant $AB$ turns about the point $A$ in the direction indicated by the arrow. The point $B$ moves closer to the point $A$. When $B$ finally coincides with $A$, the line assumes the position $XY$.




$XY$ is called a tangent to the circle.

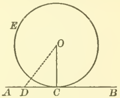
A **tangent to a circle** is a straight line which touches the circle at only one point.

The **circle** is also said to be **tangent to the line.**

The point where the tangent touches the circle is called the **Point of Tangency**, or **Point of Contact**.

## Proposition XIII. Theorem

**198.** *A straight line perpendicular to a radius at its outer extremity is a tangent to the circle.*




**Hypothesis.** $OC$ is a radius of $\odot O$; $AB \perp OC$ at $C$.

**Conclusion.** $AB$ is tangent to $\odot O$.

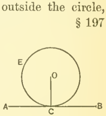
**Proof.** 1. Let $D$ be any point in $AB$ except $C$.

2. Draw $OD$.

3. $\therefore OD > OC$. Why?

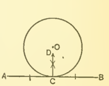
4. $\therefore$ point $D$ lies outside of the $\odot$. § 175, *c*

5. $\therefore$ every point in $AB$ except $C$ lies outside the circle, and hence $AB$ is tangent to the circle. § 197




**199. Cor.** 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Since all points in $AB$ except $C$ lie outside the circle, $OC$ is the shortest segment to $AB$ from $O$. Hence $OC \perp AB$.



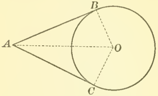

**200. Cor.** 2. *A line perpendicular to a tangent at its point of contact passes through the center of the circle.*

By Cor. 1, the radius $OC$ is $\perp$ to $AB$. Hence $OC$ and $CD$ must coincide. (Why?) $\therefore$ $CD$ passes through the center of the circle.

**201. Cor.** 3. *A line from the center of the circle perpendicular to a tangent passes through the point of contact.*

## Proposition XIV. Theorem

**202.** *The tangents to a circle from an outside point are equal.*




**Hypothesis.** $AB$ and $AC$ are tangents to $\odot O$.

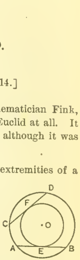
**Conclusion.** $AB = AC$.

[Proof to be given by the pupil. Recall § 114.]

**Note.** — The proof of Prop. XIV is attributed to a mathematician Fink, with the date 1583. The theorem does not appear in Euclid at all. It appears first as a definite theorem in writings of Hero, although it was apparently used by Archimedes.

**Ex. 35.** Prove that the tangents to a circle at the extremities of a diameter are parallel.

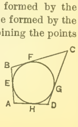
**Ex. 36.** If two circles are concentric, any two chords of the greater which are tangents of the smaller are equal.



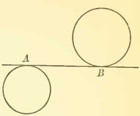

**Ex. 37.** Prove that all tangents drawn from the larger of two concentric circles to the smaller are equal.

**Ex. 38.** Prove that the line joining the center of a circle to the point of intersection of two tangents: ($a$) bisects the angle formed by the radii drawn to the points of contact; ($b$) bisects the angle formed by the tangents; ($c$) bisects and is perpendicular to the chord joining the points of contact.

**Ex. 39.** Prove that the sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two opposite sides.

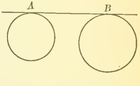


**203.** A straight line tangent to each of two circles is called a **Common Tangent** of the circles; as $AB$.




If the circles lie on opposite sides of $AB$, $AB$ is called a common internal tangent.

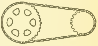
If the circles lie on the same side of $AB$, $AB$ is called a common external tangent.




The length of a common tangent is the length of the segment between the two points of contact.

Some uses of common tangents are pictured in the figures below.

BELTS AROUND PULLEYS



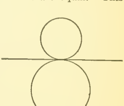
CHAIN AROUND WHEELS

**Ex. 40.** Prove that the common internal tangents of two circles are equal.

**Ex. 41.** Prove that the common external tangents of two circles are equal, when the circles are unequal.

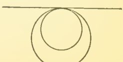
**Note.** — The theorem is also true when the circles are equal. This might be solved as an optional exercise.

**204.** Two **circles** are **tangent** when they are tangent to the same straight line at the same point.



They are tangent externally if they lie on opposite sides of the common tangent.

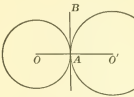
They are tangent internally if they lie on the same side of the tangent.



**Note.** — Supplementary Exercises 1–6, p. 283, can be studied now.

## Proposition XV. Theorem

**205.** *If two circles are tangent to each other, their line of centers passes through the point of contact.*




**Hypothesis.** $\circledS$ $O$ and $O'$ are both tangent to $AB$ at $A$. $OO'$ is the line of centers.

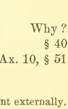
**Conclusion.** St. line $OO'$ passes through $A$.

| | | |
|---|---|---|
| **Proof.** 1. | Draw the radii $OA$ and $O'A$. | |
| 2. | Then $OA \perp AB$ and also $O'A \perp AB$. | Why? |
| 3. | $\therefore OAO'$ is a st. line. | § 40 |
| 4. | $\therefore$ st. lines $OO'$ and $OAO'$ coincide. | Ax. 10, § 51 |
| 5. | $\therefore OO'$ passes through $A$. | |

**Note.** — The theorem has been proved for two $\circledS$ tangent externally. As an optional exercise, it is suggested that the theorem be proved when the $\circledS$ are tangent internally.

**206. Cor.** *If the distance between the centers of two circles equals the sum of their radii, the circles are tangent externally.*

For then a point $A$ can be taken on $OO'$ so that $OA =$ one radius and then $O'A =$ the other radius. A perpendicular to $OO'$ at $A$ will then be tangent to each of the circles. Hence the circles are tangent (§ 204).



**Ex. 42.** Study the adjoining figure to determine how to construct it. Construct a figure like it, having the radius of the large circle 1 in. and that of the small circles $\frac{1}{2}$ in.

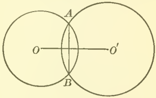
**Ex. 43.** How many common tangents do two circles have;

(*a*) Which are tangent internally?

(*b*) Which are tangent externally?

## PROPOSITION XVI. THEOREM

**207.** *If two circles intersect, the straight line joining their centers bisects their common chord at right angles.*




**Hypothesis.** ⊙ $O$ and $O'$ intersect at $A$ and $B$.

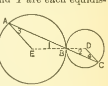
$AB$ is the common chord and $OO'$ is the line of centers.

**Conclusion.** $OO' \perp AB$ and $OO'$ bisects $AB$.

*Suggestion.* — Draw $OA$, $OB$, $O'A$, and $O'B$. (Apply § 77.)

**Ex. 44.** If two circles $O$ and $O'$ intersect at points $A$ and $B$, and if $OO'$ intersects ⊙ $O$ at $X$ and ⊙ $O'$ at $Y$, then $X$ and $Y$ are each equidistant from $A$ and $B$.

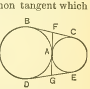
**Ex. 45.** If a straight line be drawn through the point of contact of two circles which are tangent externally, terminating in the circles, the radii drawn to its extremities are parallel.




NOTE. — The theorem is stated for two circles which are tangent externally. Investigate its truth for two circles which are tangent internally.

**Ex. 46.** If two circles are tangent to each other externally at point $A$, the tangents to them from any point in their common tangent which passes through $A$ are equal.

**Ex. 47.** If two circles are tangent to each other externally at point $A$, the common tangent which passes through $A$ bisects the other two common tangents.

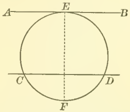



**Ex. 48.** $AB$ and $AC$ are the tangents to a circle from point $A$, and $D$ is any point in the smaller of the arcs subtended by the chord $BC$. If a tangent to the circle at $D$ meets $AB$ at $E$, and $AC$ at $F$, prove the perimeter of $\triangle\, AEF = AB + AC$.

## Proposition XVII. Theorem

**208.** *Parallel lines intercept equal arcs on a circle.*

Case I. *When one line is a tangent and one a secant:*




**Hypothesis.** $AB$ is tangent to $\odot\, CED$ at $E$; secant $CD \parallel AB$.

**Conclusion.** $\overset{\frown}{CE} = \overset{\frown}{DE}$.

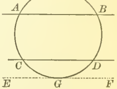
**Proof.** 1. Draw diameter $EF$.

2. $\therefore EF \perp AB$. Why?

3. $\therefore EF \perp CD$. Why?

4. $\therefore \overset{\frown}{CE} = \overset{\frown}{DE}$. § 189

Case II. *When both lines are secants:*



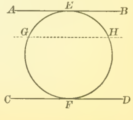

**Hypothesis.** $AB$ and $CD$ are $\parallel$ secants of $\odot\, ABDC$.

**Conclusion.** $\overset{\frown}{AC} = \overset{\frown}{BD}$.

**Proof.** 1. Assume $EGF$ tangent to the $\odot$ at $G$, and parallel to $CD$.

(Complete the proof. Compare $\overset{\frown}{AG}$ and $\overset{\frown}{BG}$; also $\overset{\frown}{CG}$ and $\overset{\frown}{DG}$.)

Case III. *When both lines are tangent:*




**Hypothesis.** $AEB$ and $CFD$ are $\parallel$ tangents to the $\odot$ at $E$ and $F$.

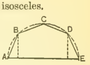
**Conclusion.** $\overset{\frown}{EGF} = \overset{\frown}{EHF}$.

[Proof to be given by the pupil.]

**Ex. 49.** Prove that the straight line joining the points of contact of two parallel tangents of a circle is a diameter of the circle.

**Ex. 50.** Prove that an inscribed trapezoid must be isosceles.

**Ex. 51.** The adjoining figure gives the method of construction of one form of *mansard roof*. The chords $AB$, $BC$, $CD$, and $DE$ are equal.



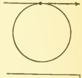

($a$) Construct such a figure for a roof whose span $AE$ is $28'$, using the scale $\frac{1}{8}'' = 1'$.

($b$) Is the line $BD$ parallel to $AE$? Prove it.

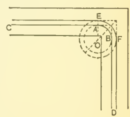
**209. Theorems concerning tangents and tangent circles have unusually wide application in design.**

Direction is naturally indicated by a straight line.

On a circle, the direction is constantly changing. It is convenient in both pure and applied mathematics to speak of the direction of a curve at a point; also it is agreed that this direction shall be the same as the direction of the tangent to the curve at the point. It is this fact which is used in a variety of ways.




If a road turns a corner as pictured, there is an abrupt change of direction. If a street car line runs along the road, such an abrupt change in direction in the tracks is impossible. For that reason, the arrangement of tracks indicated in the adjoining figure is employed. A car running from $C$ towards $D$ passes readily from $CA$ to the arc $AB$, for on both the straight line and the arc the direction at $A$ is tbe direction of line $CA$; similarly at $B$.




**Ex. 52.** Where must the center $O$ be located in order that the circle will be tangent to both $CA$ and $BD$ iu the last figure for § 209?

**Ex. 53.** Wbat kind of circles should circles $AB$ and $EF$ be?

**Ex. 54.** Determine how to construct the figures at the right. Construct such figures in circles witb diameter 3 in.

## MEASUREMENT OF ANGLES AND ARCS

**210**. To measure a given magnitude, two steps are necessary.

(*a*) Select a quantity of the same kind to be used as the unit of measure.

(*b*) Determine the number of times the given magnitude contains the unit of measure. This number is called the **Numerical Measure** of the quantity in terms of the unit employed.

If the quantity contains the unit itself or any part of it an integral number of times, the quantity can be *measured exactly*.

If the quantity does not contain the unit of measure an integral number of times, the quantity can be *measured* only *approximately*.

Thus, the diagonal of a square whose side is 1 in. is known to be 1.414 + in., where the decimal is a "never ending" decimal.

**211**. Two magnitudes of the same kind are said to be **Commensurable** when each contains the same unit of measure, called a **Common Measure**, an *integral* number of times.

Thus, two segments whose lengths are $2\frac{1}{2}$ in. and $3\frac{1}{4}$ in. respectively are commensurable, for the common measure $\frac{1}{4}$ in. is contained in the first segment 10 times and in the second 13 times.

Two magnitudes of the same kind are said to be **Incommensurable** when no unit of measure can be found which is contained an *integral* number of times in each.

The diagonal and the side of a square are incommensurable.

**212**. The **Ratio** of two magnitudes of the same kind is the quotient of their numerical measures in terms of a common measure.

Thus, the segments of lengths $2\frac{1}{2}$ in. and $3\frac{1}{4}$ in., in § 211, have the ratio $\frac{10}{13}$.

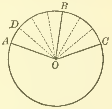
**Ex. 55**. What is the measure of a yard in terms of the unit: (*a*) 1 ft.? (*b*) 1 in.? (*c*) $\frac{1}{2}$ in.?

**Ex. 56**. What is the measure of 1 gallon in terms of the unit: (*a*) 1 qt.? (*b*) 1 pt.? (*c*) 1 gill?

**Ex. 57**. What is the ratio of 2 yd. to $1\frac{1}{2}$ ft.?

## PROPOSITION XVIII. THEOREM

**213.** *In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.*




CASE I. *When the angles are commensurable:*

**Hypothesis.** In $\odot O$, $\angle AOB$ and $\angle BOC$ are commensurable.

**Conclusion.** $$\frac{\angle AOB}{\angle BOC}=\frac{\widehat{AB}}{\widehat{BC}}.$$

**Proof.** 1. $\angle AOB$ and $\angle BOC$ have a common measure. § 211

2. Let the common measure be $\angle AOD$, and let it be contained in $\angle AOB$ 4 times and in $\angle BOC$ 3 times.

3. $$\therefore \frac{\angle AOB}{\angle BOC}=\frac{4}{3}.$$ § 212

4. The radii drawn from $O$ in step (2) divide $\widehat{AB}$ into 4 and $\widehat{BC}$ into 3 arcs which are all equal. Why?

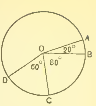
5. $$\therefore \frac{\widehat{AB}}{\widehat{BC}}=\frac{4}{3}.$$ § 212

6. $\therefore$ from steps (3) and (5),
$$\frac{\angle AOB}{\angle BOC}=\frac{\widehat{AB}}{\widehat{BC}}.$$ Ax. 1, 51

CASE II. *When the angles are incommensurable:*

The theorem is true also in this case. The proof presents certain difficulties which it is wise to postpone at this time. This proof is taken up in § 423.

**Ex. 58.** In the adjoining figure, compare $\widehat{AB}$ and $\widehat{BC}$. Also compare $\widehat{AB}$ and $\widehat{DC}$; also $\widehat{BC}$ and $\widehat{DC}$.




**Ex. 59.** A right central angle is what part of a straight angle? What part therefore is its intercepted arc of a semicircle?

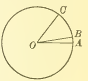
**Ex. 60.** A 60° angle is what part of the perigon? What part therefore is its intercepted arc of the whole circle?

**Ex. 61.** If a circle is divided into 5 equal parts, what part of the perigon is the central angle which intercepts one of the parts? How many degrees are there in the central angle?

**Ex. 62.** If $AB$, any chord of circle $O$, is extended to a point $C$ so that $BC$ equals the radius of the circle, and $CO$ is drawn, cutting the circle at $Z$ and $E$ respectively, then $\widehat{AE} = 3 \cdot \widehat{BZ}$.

*Suggestions.*—1. Draw $OA$ and $OB$. 2. Prove $\angle AOE = 3\angle BOC$. Use § 87 and § 69.

**214. Measuring Angles and Arcs.** In § 28, the unit for measuring angles is given as 1 degree, $\frac{1}{90}$ of a right angle or $\frac{1}{360}$ of the perigon. This will be called for the present one *angular-degree.* Let $\angle AOB$ represent 1°. Similarly, we shall speak of *angular-minutes* and *angular-seconds.* Thus, 60 angular-seconds equal one angular-minute; and 60 angular-minutes equal one angular-degree.




Let a circle be drawn around point $O$ as center, and the radii which divide the perigon into 360 equal central angles be imagined. These angles are angular-degrees. They will intercept 360 equal arcs on the circle. Let $\widehat{AB}$ represent one of these arcs. It is the unit for measuring arcs *on this circle and on any equal circle.* It will be called one arc-degree.

Evidently on a circle with longer radius, the arc corresponding to $\widehat{AB}$ will be longer.

In similar manner, each arc-degree could be divided into 60 equal parts, called arc-minutes, and each arc-minute into 60 equal arc-seconds. A central angle of one angular minute intercepts an arc of one arc-minute.

**215. A central angle has the same measure as its intercepted arc,** *when angular-degrees and arc degrees are used as the respective units of measure.*

Let $\angle AOB$ represent 1 angular-degree and $\angle AOC$ any other central angle.

Then $$\frac{\angle AOC}{\angle AOB} = \frac{\overset{\frown}{AC}}{\overset{\frown}{AB}}. \qquad \S\ 213$$

But $\dfrac{\angle AOC}{\angle AOB}$ is the numerical measure $\angle AOC$, and $\dfrac{\overset{\frown}{AC}}{\overset{\frown}{AB}}$ is the numerical measure of $\overset{\frown}{AC}$, by the definition.

Hence the measure of $\angle AOC$ equals the measure of $\overset{\frown}{AC}$.

Thus, if $\angle AOC = 57.29$ angular-degrees, then $\overset{\frown}{AC} = 57.29$ arc-degrees.

From now on, it will be understood that angles are measured in terms of angular-degrees, and arcs in terms of arc-degrees. Also, the following statement of the theorem of § 215 will be employed for convenience:

*A central angle is measured by its intercepted arc.*

**Ex. 63.** What is an arc-degree? An angular-degree?

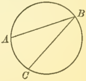
**Ex. 64.** Are all angular-degrees of the same size?

Are all arc-degrees of the same size: ($a$) on the same or on equal circles? ($b$) on unequal circles?

**Ex. 65.** If $ABCD$ is an inscribed square and $O$ is the center of the circle, how may degrees are there in $\overset{\frown}{AB}$? in $\angle AOB$?

**Ex. 66.** $\triangle ABC$ is an equilateral triangle inscribed in a circle with center $O$; how many degrees are there in $\overset{\frown}{AB}$? in $\angle AOB$?

**216.** An angle is said to be an **Inscribed Angle** when its vertex is on the circle and its sides are chords of the circle; as $\angle ABC$.



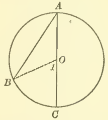

$\angle ABC$ intercepts the $\overset{\frown}{AC}$; $\overset{\frown}{AC}$ is intercepted by the $\angle ABC$.

Such an angle is said to be inscribed in a circle or may be said to be inscribed in the arc $ABC$.

## Proposition XIX. Theorem

**217.** *An inscribed angle is measured by one half its intercepted arc.*

Case I. *When one side of the angle is a diameter:*



**Hypothesis.** $AC$ is a diameter; $AB$ is any other chord of $\odot O$.

**Conclusion.** $\angle BAC$ is measured by $\frac{1}{2}\ \widehat{BC}$.

**Proof.** 1. Draw $BO$.

2. $\angle 1 = \angle B + \angle A$. Why?

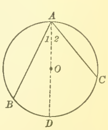
3. $\angle B = \angle A$. Why?

4. $\therefore \angle 1 = 2 \cdot \angle A$, or $\angle A = \frac{1}{2} \angle 1$.

5. But $\angle 1$ is measured by $\widehat{BC}$. § 215

6. $\therefore \angle A$ is measured by $\frac{1}{2}\ \widehat{BC}$.

Case II. *When the center of the $\odot$ is within the angle:*



**Hypothesis.** Center $O$ lies within inscribed $\angle BAC$.

**Conclusion.** $\angle BAC$ is measured by $\frac{1}{2}\ \widehat{BC}$.

**Proof.** 1. Draw diameter $AD$.

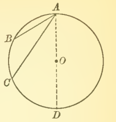
2. Then $\angle 1$ is measured by $\frac{1}{2}\ \widehat{BD}$,

and $\angle 2$ is measured by $\frac{1}{2}\ \widehat{DC}$. Case I

3. $\therefore \angle 1 + \angle 2$ is measured by $\frac{1}{2}\ (\widehat{BD} + \widehat{DC})$, Ax. 3, § 51

or $\angle BAC$ is measured by $\frac{1}{2}\ \widehat{BC}$.

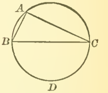
Case III. *When the center of the ⊙ lies outside the angle:*




**Hypothesis.** Center $O$ lies outside inscribed $\angle BAC$.

**Conclusion.** $\angle BAC$ is measured by $\frac{1}{2}\,\widehat{BC}$.

*Suggestions.* — 1. $\angle BAD$ is measured by what? 2. $\angle CAD$?

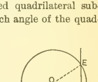



**218. Cor. 1.** *An angle inscribed in a semicircle is a right angle.*

(If $BC$ is a diameter, prove $\angle A = 1$ rt. $\angle$.)

**219. Cor. 2.** *Inscribed angles which intercept the same arc are equal.*

**Ex. 67.** Three consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 90°, and 60° respectively. Find each angle of the quadrilateral.

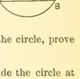



**Ex. 68.** Construct a line perpendicular to a given segment at one extremity of the segment.

Take $O$ any point not in segment $AB$ and draw a circle with $O$ as center and $OB$ as radius, cutting $AB$ at $D$. Draw $DO$ meeting the circle at $E$. Then $EB \perp AB$ at $B$. Prove it.

**Ex. 69.** If chords $AB$ and $CD$ intersect at $E$ within the circle, prove that $\triangle AEC$ and $\triangle BDE$ are mutually equiangular.

**Ex. 70.** If chords $AB$ and $CD$ extended meet outside the circle at point $E$, prove $\triangle ADE$ and $\triangle BCE$ are mutually equiangular.



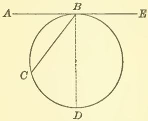

**Ex. 71.** Prove that the opposite angles of an inscribed quadrilateral are supplementary.

($\angle B$ is measured by what? $\angle D$? $\therefore \angle B + \angle D$?)

**Note.** — Supplementary Exercises 9 to 20, p. 284, can be studied now.

## Proposition XX. Theorem

**220.** *The angle formed by a tangent and a chord drawn to the point of contact is measured by one half its intercepted arc.*



**Hypothesis.** $AE$ is tangent to $\odot\, CBD$ at $B$; $BC$ is a chord.

**Conclusion.** $\angle\, ABC$ is measured by $\frac{1}{2}\, \widehat{BC}$.

**Proof.** 1. Draw diameter $BD$; then $BD \perp AE$. Why?

2. $\angle\, ABD = 90°$, and $\widehat{BCD} = 180°$. Why?

3. $\therefore\ \angle\, ABD$ is measured by $\frac{1}{2}\, \widehat{BCD}$.

4. $\angle\, CBD$ is measured by $\frac{1}{2}\, \widehat{CD}$. Why?

5. $\therefore\ \angle\, ABD - \angle\, CBD$ is measured by $\frac{1}{2}\, \widehat{BCD} - \frac{1}{2}\, \widehat{CD}$. Ax. 4, § 51

6. $\therefore\ \angle\, ABC$ is measured by $\frac{1}{2}\, \widehat{BC}$.

7. Similarly, $\angle EBC$ is measured by $\frac{1}{2}\, \widehat{BDC}$.

**Ex. 72.** If, in the figure for Prop. XX, $\widehat{BC} = 110°$, how many degrees are there in $\angle\, ABC$ and $\angle\, EBC$?

**Ex. 73.** If tangents are drawn to a circle at the extremities of a a chord, they make equal angles with the chord.

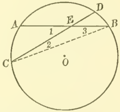
**Ex. 74.** If two tangents drawn from a point to a circle form an angle of 60°, then each of the tangents equals the chord joining the points of contact. (Prove the triangle is equilateral.)

**Ex. 75.** If a tangent be drawn to a circle at the extremity of a chord, the line joining the mid-point of the intercepted arc to the point of contact bisects the angle formed by the tangent and the chord.

**Ex. 76.** Prove that a tangent to a circle at the mid-point of an arc is parallel to the chord of the arc. (§ 93.)

## Proposition XXI. Theorem

**221.** *The angle formed by two chords intersecting within a circle is measured by one half the sum of the arcs intercepted by it and its vertical angle.*




**Hypothesis.** Chords $AB$ and $CD$ intersect at $E$ within $\odot O$.

**Conclusion.** $\angle 1$ is measured by $\frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{DB})$.

| | | |
|---|---|---|
| **Proof.** 1. | Draw $CB$. | |
| 2. | $\angle 1 = \angle 3 + \angle 2$. | Why? |
| 3. | $\angle 3$ is measured by $\frac{1}{2}\overset{\frown}{AC}$. | Why? |
| 4. | $\angle 2$ is measured by $\frac{1}{2}\overset{\frown}{DB}$. | Why? |
| 5. | $\therefore \angle 1$ is measured by $\frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{DB})$. | Ax. 3, § 51 |

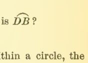
**Ex. 77.** If $\overset{\frown}{AC} = 70°$ and $\overset{\frown}{DB} = 50°$, how many degrees are there in $\angle AEC$?

**Ex. 78.** If $\overset{\frown}{AC} = 74°$ and $\angle AEC = 50°$, how large is $\overset{\frown}{DB}$?

*Suggestion.* — Let $\overset{\frown}{DB} = x°$.

**Ex. 79.** If two chords intersect at right angles within a circle, the sum of one pair of opposite intercepted arcs is equal to a semicircle.

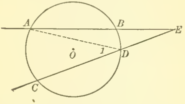
**Ex. 80.** If $\overset{\frown}{AX} = \overset{\frown}{CY}$ and $\overset{\frown}{AB} = \overset{\frown}{CB}$, prove $\triangle MNB$ is an isosceles triangle.




**Note.** — Supplementary Exercises 21 to 23, p. 286, can be studied now

## Proposition XXII. Theorem

**222.** *The angle formed by two secants intersecting outside the circle is measured by one half the difference between its intercepted arcs.*




**Hypothesis.** Secants $AB$ and $CD$ intersect at $E$ outside $\odot\, O$.

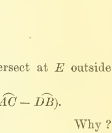
**Conclusion.** $\angle E$ is measured by $\frac{1}{2}(\widehat{AC} - \widehat{DB})$.

**Proof.** 1. $\angle 1 = \angle A + \angle E$. Why?

2. $\therefore\ \angle E = \angle 1 - \angle A$. By algebra.

[Complete the proof. Obtain the measures of $\angle 1$ and $\angle A$ and then determine the measure of $\angle E$.]

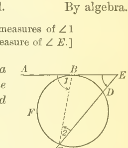
**223. Cor. 1.** *The angle formed by a secant and a tangent is measured by one half the difference between its intercepted arcs.*




Prove $\angle E$ is measured by

$\frac{1}{2}(\widehat{BC} - \widehat{DB})$.

**224. Cor. 2.** *The angle formed by two tangents is measured by one half the difference between its intercepted arcs.*




Prove $\angle E$ is measured by

$\frac{1}{2}(\widehat{BFD} - \widehat{BGD})$.

**Ex. 81.** If in § 222 $\widehat{AC} = 100°$ and $\widehat{BD} = 40°$, how large is $\angle E$?

**Ex. 82.** If in § 222 $\widehat{AC}$ is a quadrant, and $\angle E$ is 40°, how large is $\widehat{BD}$?

*Suggestion.* — Let $BD = x°$.

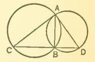
**Ex. 83.** If $\widehat{AC}$ in the figure of Prop. XXII is 120°, and $\angle A = 15°$, how large is $\angle E$?

**Ex. 84.** If in the figure for § 223 $\angle E = 50°$ and $\widehat{BD} = 70°$, how how large is $\widehat{BFC}$?

**Ex. 85.** If in the figure for § 224, $\widehat{BFD} = \frac{4}{9}$ of the circle, how large is $\angle E$?

**Ex. 86.** If in § 223 $\widehat{BFC} = 100°$, and $\widehat{CD} = 200°$, how many degrees are there in angle $E$?

**Ex. 87.** If $AB$ is the common chord of two intersecting circles, and $AC$ and $AD$ are diameters drawn from $A$, prove that line $CD$ passes through $B$.



*Suggestion.* — Draw $CB$ and $BD$, and try to prove $CBD$ a straight line.

**Ex. 88.** A square $ABCD$ is inscribed in a circle. A tangent is drawn to the circle at point $A$. How large is the angle formed by the tangent and side $AB$?

**Ex. 89.** The line joining the mid-points of the arcs subtended by the sides $AB$ and $AC$ of inscribed $\triangle\ ABC$ cuts $AB$ at $F$ and $AC$ at $G$. Prove $AF = AG$.

**Ex. 90.** If $AB$ and $AC$ are two chords of a circle making equal angles with the tangent at $A$, prove $AB = AC$.

**Ex. 91.** If $ABCD$ is an inscribed quadrilateral, and $AD$ and $BC$ extended meet at $P$, the tangent $XY$ at $P$ to the circle circumscribed about the $\triangle\ ABP$ is parallel to $CD$.

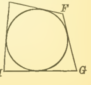
*Suggestions.* — 1. $XY \parallel CD$ if $\angle DCP = ?$

2. Compare each of these angles with $\angle BAD$. Recall Ex. 73.

**Note.** — Supplementary Exercises 24 to 30, p. 286, can be studied now.

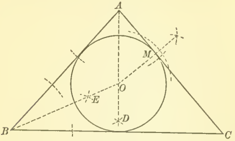
**225.** A **circle** is said to be **inscribed in a polygon** when it is tangent to each side of the polygon.

The **polygon** is said to be **circumscribed about the circle**; as $EFGH$.



## PROPOSITION XXIII. PROBLEM

**226.** *Inscribe a circle in a given triangle.*




**Given** $\triangle ABC$.

**Required** to inscribe a $\odot$ in $\triangle ABC$.

**Construction.** 1. Construct the bisectors $BE$ and $AD$ of $\angle B$ and $\angle A$ respectively, meeting at point $O$.

2. Construct $OM \perp AC$.

3. With $O$ as center and $OM$ as radius, draw a $\odot$.

**Statement.** This circle will be tangent to $AB$, $BC$, and $AC$.

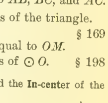
**Proof.** 1. $O$ is equidistant from the sides of the triangle. § 169

2. $\therefore$ ⊥s from $O$ to the sides are all equal to $OM$.

3. $\therefore AB$, $BC$, and $AC$ are tangents of $\odot O$. § 198

**Note 1.** — Point $O$ is the point which was called the **In-center** of the triangle in § 169. The reason is clear now.

**Note 2.** — A circle can be constructed which is tangent to the sides $AB$ and $AC$ prolonged and to $BC$ as in the adjoining figure. It is called an **Escribed Circle** and its center is called an **Ex-center** of the triangle.



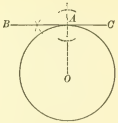

There are three ex-centers for each triangle.

**Ex. 92.** Construct a triangle and construct its three escribed circles.

**Ex. 93.** If $O$ is the center of the circumscribed circle of $\triangle ABC$ and $OD$ is drawn perpendicular to $BC$, prove $\angle BOD = \angle A$.

## Proposition XXIV. Problem

**227.** I. *Construct a tangent to a circle at a point on the circle.*



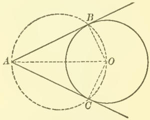

**Given** ⊙ $O$ and point $A$ on it.

**Required** to construct a tangent to ⊙ $O$ at $A$.

Construction indicated in the figure.

[Description and proof to be given by the pupil.]

II. *Construct a tangent to a circle from a point outside the circle.*




**Given.** ⊙ $O$ and point $A$ outside ⊙ $O$.

**Required** to construct a tangent to ⊙ $O$ from point $A$.

**Construction.** 1. Draw $AO$.

2. Construct a ⊙ on $AO$ as diameter intersecting ⊙ $O$ at $B$ and $C$.

3. Draw $AB$ and $AC$.

**Statement.** $AB$ and $AC$ are both tangent to ⊙ $O$.

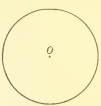
[Proof to be given by the pupil.]

*Suggestion.*—Draw $OB$ and $OC$. Prove $\angle B$ and $\angle C$ are rt. $\angle$s.

## LOCI

**228.** Illustrative Problem 1. — Where are all points $\frac{1}{2}$ in. from $O$?

Evidently the *place* of points $\frac{1}{2}$ in. from $O$ is the circle with center $O$ and radius $\frac{1}{2}$ in.




Instead of using the word "place" it is customary to use the word *locus* — a Latin word meaning place. So the preceding sentence becomes

The locus of points $\frac{1}{2}$ in. from $O$ is the circle with center $O$ and radius $\frac{1}{2}$ in.

It is evident that:

(*a*) Every point "$\frac{1}{2}$ in. from $O$" is on the circle.

(*b*) Every point on the circle is "$\frac{1}{2}$ in. from $O$."

"$\frac{1}{2}$ in. from $O$" is the condition which the points satisfy.

**Ex. 94.** Draw the locus of points which are 2 in. from a given point.

**Ex. 95.** Draw the locus of the end of a pump handle which is 33 in. long from its end to the point about which it turns, if the handle may be moved through an angle of 100°. (Let 1 in. represent 11 in.)

**Ex. 96.** Draw any line of indefinite length.

(*a*) Locate freehand three points above the line which are 1 in. from the line.

(*b*) Draw the line which contains all points which are 1 in. from the line and lie above the line.

(*c*) Are there any other points which are 1 in. from the line?

(*d*) Draw the line showing where they are to be found.

**Ex. 97.** Where are, that is, *what is the locus of* all points on this page which are $\frac{1}{2}$ in. from the left-hand edge of the page?

**Ex. 98.** A rectangular lot is 100 ft. wide and 300 ft. long. Shrubs are to be planted 5 ft. from the lot line along the two sides and the back of the lot. What is the locus of the shrubs?

**Ex. 99.** Draw two parallel lines.

(*a*) Locate freehand three points which are equidistant from the two parallels.

(*b*) Draw the locus of all points which are equidistant from the parallels.

**Ex. 100.** (*a*) Draw a line $AB$ and locate on it a point $C$. Construct three circles, all tangent to $AB$ at $C$.

(*b*) What is the locus of the center of a ⊙ which will be tangent to a given line at a given point?

**Ex. 101.** Draw a circle with radius 1 in.

Draw five radii of the circle. On each radius locate a point $\frac{1}{4}$ in. from the circle. Draw the locus of all such points.

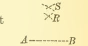
**Ex. 102.** Draw the locus of all points outside a circle with 1 in. radius and $\frac{1}{4}$ in. from the circle.

**229. Def.** If a single geometrical condition is given, the **Locus of Points** satisfying that condition is the line or group of lines such that:

(*a*) Every point in the line (or lines) satisfies the condition.

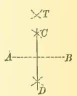
(*b*) Every point which satisfies the condition lies in the line or group of lines.

**230. Problem.** *Determine the locus of points equidistant from two given points.*




**Solution.** (*a*) 1. $R$ is located so that $RA = RB$. Similarly, $S$ and $T$ are located.

2. Their position suggests that the locus of such points is the ⊥ bisector of $AB$.




(*b*) 1. Assume that $CD$, the ⊥ bisector $AB$, is the locus of points equidistant from $A$ and $B$.

2. Is every point on $CD$ equidistant from $A$ and $B$?

Yes, by § 118, I.

3. Is every point equidistant from $A$ and $B$ in line $CD$?

Yes, by § 118, II.

(*c*) ∴ *The locus of points equidistant from two given points is the perpendicular-bisector of the segment between the points.*

**Note.** — In solving a locus problem, first locate three or more points satisfying the condition; then decide what you think the locus is; then try to prove that the supposed locus is the required locus. In doing this last, prove theorems (*a*) and (*b*) of § 229, as is done in part (*b*) of the solution in § 230.

**Ex. 103.** Locate two points $X$ and $Y$ which are 2 in. apart. Construct the locus of points equidistant from $X$ and $Y$.

**Ex. 104.** What is the locus of the vertex of an isosceles triangle which has a given base?

**Ex. 105.** What is the locus of the center of a circle which will pass through two given points?

**231. Problem.** *Determine the locus of points within an angle which are equidistant from the sides of the angle.*

[The solution is to be given by the pupil.]

*Suggestions.* — Model your solution after that for the problem in § 230. Recall § 120, I and II.

At the end of your solution complete the following sentence:

The locus of points within an angle which are equidistant from the sides of the angle is . . . .

**Ex. 106.** Construct the locus of points equidistant from the sides of a right angle and within the angle.

**Ex. 107.** What is the locus of the center of a circle which is tangent to the sides of a given angle and lies within the angle?

**Note.** — For additional discussion of loci see § 238.

## CONSTRUCTION OF TRIANGLES

**232.** In a $\triangle ABC$, the sides opposite angles $A$, $B$, and $C$ are marked by the small letters $a$, $b$, and $c$, respectively.


The letter $h$ denotes an altitude. $h_a$ (read $h$ – sub – $a$) denotes the altitude to side $a$. Similarly, there are the altitudes $h_b$ and $h_c$.

The letter $m$ denotes a median. The medians to sides $a$, $b$, and $c$ are denoted by $m_a$, $m_b$, and $m_c$ respectively.

The letter $t$ is used to denote the length of the bisector of an angle between the vertex and the opposite side. The bisectors of angles $A$, $B$, and $C$ are denoted by $t_A$, $t_B$, and $t_C$.

**Note.** — This notation was introduced by Euler (1707–1783).

**233.** A triangle is determined when three independent parts are known.

### General Suggestions

1. *Draw freehand a triangle which represents the desired figure, marking with heavy lines the parts which correspond to the given parts. Use this figure as a guide in constructing the desired triangle. It is not the desired triangle, and the parts marked are not necessarily equal in size to the given parts.*

2. *Make the construction, using the given parts.*

3. *Prove that the resulting triangle has all the given parts, and is the kind of triangle specified.*

4. *Discuss the construction, determining whether there are conditions under which it may be impossible to construct a triangle having the given parts.* (*See Prop. IV, Book I, Discussion.*)

**234.** The following seven problems are the fundamental construction problems for triangles:

**Ex. 108.** Review Proposition IV, Book I.

**Ex. 109.** Construct a triangle having given two of its sides and the included angle.

*Suggestion.* — Let $a$ and $b$ be two given segments and $\angle C$ a given $\angle$, — drawn at random; then construct the triangle.

**Ex. 110.** Construct a triangle having given two of its angles and the included side.

**Discussion.** — Can the triangle be constructed always:

($a$) If both $\angle$s are acute? ($b$) If both $\angle$s are rt. $\angle$s? ($c$) If both $\angle$s are obtuse? ($d$) If one $\angle$ is obtuse and one is acute?

**Ex. 111.** Construct a right triangle having given its hypotenuse and a leg.

**Ex. 112.** Construct a triangle having given a side, the opposite angle, and another angle.

*Suggestion.* — If $a$, $\angle B$, and $\angle A$ are given, then $\angle C$ may be determined by subtracting $\angle A + \angle B$ from 180°. Then $\triangle ABC$ can be constructed.

**Ex. 113.** Construct a right triangle having given a leg and the opposite acute angle.

**Ex. 114.** Construct a right triangle having given the hypotenuse and an acute angle.

Note. — For further discussion of construction of figures see § 235 and § 241.

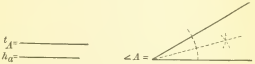
## SUPPLEMENTARY TOPICS

**Note.** — The rest of Book II is supplementary material and may safely be omitted.

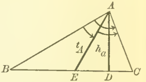
**235.** Triangles may be constructed when numerous other combinations of three independent parts are given, besides those mentioned already in § 234.

Illustrative Problem. — Construct a triangle having given an angle, the length of its bisector, and the length of the altitude drawn from its vertex.

**Given**




**Required** to construct $\triangle ABC$.



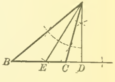

**Analysis.** 1. Let $\triangle ABC$, with $AD \perp BC$ and $AE$ bisecting $\angle A$ represent the required figure.

2. The known parts are marked with heavy lines, including $\angle BAE = \angle EAC = \frac{1}{2}\angle A$.

3. $\triangle ADE$ is a rt. $\triangle$ with a known leg ($= h_a$) and known hypotenuse ($= t_A$). Hence $\triangle ADE$ can be constructed. (Ex. 111.)

4. $B$ is on $DE$ extended and $\angle BAE = \frac{1}{2}\angle A$.

5. $C$ is on $ED$ extended and $\angle EAC = \frac{1}{2}\angle A$.




**Construction.** 1. Construct rt. $\triangle ADE$ with leg $= h_a$ and hypotenuse $= t_A$.

2. Extend $DE$ in both directions.

3. Bisect $\angle A$, and construct $AB$, making $\angle EAB = \frac{1}{2}\angle A$; let $AB$ meet $DE$ extended at $B$.

4. Construct $AC$, making $\angle EAC = \frac{1}{2}\angle A$, and meeting $DE$ extended at $C$.

5. Then $\triangle ABC$ is the required triangle.

**Proof.** 1. $\angle BAC =$ given $\angle A$, since it equals $2(\frac{1}{2}\angle A)$. Const.

2. $AD =$ given $h_a$ and is an altitude since $\angle D =$ rt. $\angle$. Const.

3. $AE =$ given $t_A$ and is the bisector of $\angle A$. Const.

**Discussion.** The construction is impossible if $t_A < h_a$.

**Note.** — Observe that the final triangle may appear quite different from the triangle drawn for the first step in the analysis.

### 236. Analysis of Construction Problems.

1. Draw a figure which represents the desired figure.

(*a*) Make this figure general. For example, if a triangle is to be drawn, do not draw a right or an isosceles triangle unless such a triangle is specified.

(*b*) Remember that this is not the final figure and that the parts in it are not necessarily the given parts.

2. Mark with heavy lines or with colored lines the parts which are known and also those which may be readily determined from the known parts by fundamental constructions.

For example, if a known line is bisected, then each of the halves is known.

3. Try to determine some part of the figure which can be constructed by known methods. Usually this is a triangle. This part can usually be made the basis for the rest of the construction.

In this connection, remember the first five fundamental triangle constructions given in § 234.

4. Try to determine how the remaining parts can be obtained from the figure constructed in step 3.

5. Make the construction, following the points noted in steps 3 and 4.

6. Prove that the resulting figure satisfies the conditions of the problem.

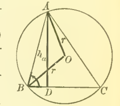
7. Discuss the resulting figure, determining, in particular, whether the construction is or is not always possible.

**Note.** — Systematic use of this form of analysis is attributed to Plato. The method of analysis has been described as one of the four great steps in mathematics. Plato also introduced the restriction that constructions should be made by ruler and compass alone.

Illustrative Problem 1. — Construct $\triangle ABC$ having given $\angle B$, $h_a$, and the radius $r$ of the circumscribed circle.

**Given**


**Required** to construct $\triangle ABC$.




**Analysis.** 1. Let the adjoining figure represent the required figure.

2. $\triangle ABD$ is a rt. $\triangle$, with known leg $(= h_a)$ and known acute angle $(\angle B)$. $\therefore$ it can be constructed by Ex. 113.

3. Point $O$ is equidistant from $A$ and $B$, a distance equal to $r$. Hence $O$ can be located.

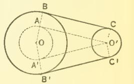
4. The circle can then be drawn, and $BD$, extended, will meet the circle at $C$.

**Construction** left to the pupil. Follow up the steps 2, 3, and 4.

1. Construct a rt. $\angle$ with side $= h_a$ and opposite $\angle = \angle B$.
2. Locate the point $O$ and draw the circle.
3. Extend $BD$ and thus determine point $C$.

Proof and Discussion left to the pupil.

Illustrative Problem 2. — Construct the common external tangents of two circles.




**Analysis.** 1. Let the circles be assumed unequal. Let the adjoining figure represent the desired figure.

2. Evidently $ABCO'$ is a $\square$.

3. $\therefore AB = CO'$.

4. $\therefore AO = OB - CO'$.

5. Also, $AO'$ and $A'O'$ are tangents to circle $AA'$.

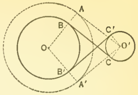
**Construction.** 1. Construct a circle with radius equal to the difference between the radii of the two circles, and concentric with the larger circle.

2. Draw tangents to this circle from the center of the smaller circle, meeting the constructed circle at points $A$ and $A'$.

3. Draw $OA$ and $OA'$ meeting the large circle at $B$ and $B'$.

Complete the construction.

Give the Proof and the Discussion.




**Ex. 115.** Construct the common internal tangents of two unequal circles.

Construct the triangle $ABC$ having given:

**Ex. 116.** $b$, $c$, $h_c$.

**Ex. 117.** $a$, $c$, and $m_c$.

**Ex. 118.** $a$, $b$, and $h_c$.

**Ex. 119.** $b$, $h_c$, $B$.

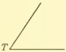
**Ex. 120.** Construct an isosceles triangle having given one base angle and the altitude to the base.

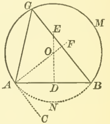
**Ex. 121.** Construct an isosceles triangle having given one side and the altitude to one of the sides.

**Note.** — Supplementary Exercises 31–49, p. 287, can be studied now.

## Proposition XXV. Problem

**237.** *Upon a given segment as chord, construct an arc of a circle such that every angle inscribed in it shall equal a given angle.*




**Given** segment $AB$ and $\angle T$.

**Required** to construct an arc upon $AB$ as chord such that every angle inscribed in the arc shall equal $\angle T$.

**Construction.** 1. Construct $\angle BAC = \angle T$.

2. Construct $DE \perp AB$ at its mid-point.

3. Construct $AF \perp AC$ at $A$, intersecting $DE$ at $O$.

4. Construct a circle with center $O$ and $OA$ as radius.

**Statement.** $\overset{\frown}{AMB}$ is the required arc.

**Proof.** (The pupil should now prove that any $\angle AGB = \angle T$.)

### FURTHER DISCUSSION OF LOCI *

**238. Method of Attacking a Locus Problem.**

1. Locate either freehand or by construction three or more points which satisfy the given condition. These points should suggest the probable locus.

2. Draw the probable locus and try to prove that it is the real locus. To do this, try to prove either (*a*) and (*b*) below, or else (*a*) and (*c*):

(*a*) Every point on the locus satisfies the given condition.

(*b*) Every point which satisfies the given condition lies on the locus.

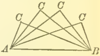
(*c*) Every point not on the locus does not satisfy the given condition.

**Note.** — (*b*) is the converse of (*a*) and (*c*) is the opposite of (*a*). The opposite of (*b*) is:

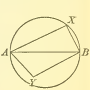
(*d*) every point which does not satisfy the given condition does not lie on the locus.

When (*a*) and (*b*) are known, then (*c*) and (*d*) can be proved by the indirect method; when (*a*) and (*c*) are known, then (*b*) and (*d*) can be proved in the same manner.

Illustrative Problem. — Determine the locus of the vertex of the right angle of a right triangle having a given segment as hypotenuse.




*Solution.* 1. Let $\triangle ABC$ be right triangles having the hypotenuse $AB$ and rt. $\angle$ at $C$.




2. This figure suggests that the points $C$ lie on a circle having $AB$ as diameter.

3. Assume that the locus is the circle having $AB$ as diameter.

(*a*) Is every $\triangle AXB$, where $X$ is any point on the circle, a rt. $\triangle$?

Yes, since $\angle AXB$ is a rt. $\angle$, by § 218.

* Review at this time § 229 and the two loci discussed in § 230 and § 231.

(*b*) Is every $\triangle AYB$, where $Y$ is any point not on the circle, an oblique $\triangle$?

Yes, since every $\angle AYB$ is either acute or obtuse, according as $Y$ lies outside or inside of the circle. (Easily proved.)

4. *Hence the locus of the vertex of the right angle of a right triangle having a given segment as hypotenuse is a circle drawn on the hypotenuse as diameter.*

**239. Summary of Fundamental Loci.**

1. The locus of points at a given distance $d$ from a given point $O$ is the circle drawn with $O$ as center and $d$ as radius. (§ 228.)

2. The locus of points at a given distance $d$ from a fixed line $l$ (of indefinite length) is the pair of parallels to $l$ at the distance $d$ from it. (Ex. 96.)

3. The locus of points equidistant from two parallel lines is the line parallel to them and midway between them. (Ex. 99.)

4. The locus of points equidistant from two given points is the perpendicular bisector of the segment joining the points. (§ 230.)

5. The locus of points equidistant from the sides of an angle and within the angle is the bisector of the angle. (§ 231.)

**Cor.** The locus of points equidistant from two intersecting straight lines is the set of bisectors of their included angles.

(These bisectors form two straight lines.)

6. The locus of the vertex of the right angle of a right triangle which has a given hypotenuse is a circle drawn upon the hypotenuse as diameter. (§ 238.)

7. If $A$ and $B$ are any two fixed points and $X$ is a point such that $\angle AXB$ is a given angle, the locus of $X$ is the arc of a circle constructed upon $AB$ as chord such that every angle inscribed in it equals the angle given. (§ 237.)

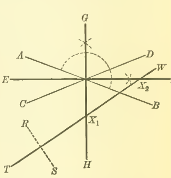
**Caution.** — 1. Remember that "what is the locus of?" means "what is the place of?" 2. Be certain that you know what "the given condition" is in each locus theorem.

**240. Intersection of Loci.** — Sometimes it is specified that a point shall satisfy each of two given conditions. Each condition determines a locus for the point. The point then must lie at the intersection of the two loci.

Illustrative Problem. — Find all points which are equidistant from two intersecting lines and also equidistant from two fixed points.

**Given** intersecting lines $AB$ and $CD$ and points $R$ and $S$.

**Required** to find all points which are equidistant from $AB$ and $CD$ and also equidistant from $R$ and $S$.




**Solution.** 1. The locus of points equidistant from $AB$ and $CD$ is the set of bisectors of the angles included by them. (Lines $EF$ and $GH$.)

2. The locus of points equidistant from $R$ and $S$ is the perpendicular bisector of $RS$. (Line $TW$.)

3. The required points will be at the intersection of $TW$ with $EF$ and $GH$.

**Discussion.** 1. Usually there are two points; as $X_1$ and $X_2$.

2. There may be only one point, however, for $TW$ may be parallel to one of the lines, $EF$ and $GH$.

3. There must always be at least one point, for $TW$ cannot be parallel to both $EF$ and $GH$.

4. There may be a whole line full of points, for $TW$ may coincide with $EF$ or $GH$.

**Ex. 122.** In a given line, find all points which are equidistant from two given points.

**Ex. 123.** In a given line, find all points which are equidistant from two given intersecting lines.

**Ex. 124.** In a given circle, find all points which are equidistant from two given parallel lines.

**Ex 125.** Find all points which are equidistant from two given points and also at a given distance from a given point.

**Ex. 126.** Find all points which are equidistant from two given points and also at a given distance from a given line.

**Ex. 127.** Find all points which are equidistant from two given points and also equidistant from two given parallels.

**Ex. 128.** Find all points which are equidistant from two given parallels and also at a given distance from a given point.

**Ex. 129.** Find all points which are at a given distance from a given line and also at another given distance from a given point.

**Ex. 130.** Find all points which are equidistant from two parallels and also equidistant from two intersecting lines.

**Ex. 131.** Find all points which are equidistant from two intersecting lines and at a given distance from a given point.

**Ex. 132.** What is the locus of the vertex of a triangle whose base lies in a given straight line if the altitude to the base is a given segment?

**Ex. 133.** What is the locus of the center of a circle which shall be tangent to a given line and have a given radius?

**Ex. 134.** What is the locus of points at a given distance from a given circle?

(The distance is measured along a line between the point and the center of the circle.)

**Ex. 135.** What is the locus of the center of a circle which has a given radius and passes through a given fixed point?

**Ex. 136.** What is the locus of the center of a circle which shall be tangent to each of two parallel lines?

**Ex. 137.** What is the locus of the mid-points of all chords of a circle that have a given length?

**Ex. 138.** What is the locus of the points such that the tangents from the points to a given circle shall have a given length?

**Ex. 139.** What is the locus of the mid-points of all parallel chords of a circle?

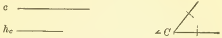
**Ex. 140.** What is the locus of the mid-points of all segments drawn from one vertex of a triangle and terminated by the opposite side?

**Ex. 141.** A line $AB$ of fixed length moves so that $A$ is constantly on one side of a given right angle and $B$ is on the other side of the angle. What is the locus of the mid-point of the segment $AB$? (Recall Ex. 175, Book I.)

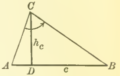
**241. Construction of Figures by Intersection of Loci.**

Illustrative Problem. — Construct $\triangle ABC$ having given $c$, $h_c$, and $\angle C$.

**Given**




**Required** to construct $\triangle ABC$.

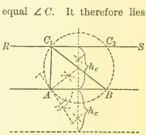



**Analysis.** 1. Let $\triangle ABC$ represent the desired triangle, the known parts being marked by heavy lines.

2. Line c can be drawn, thus locating definitely points $A$ and $B$.

3. Point $C$ is at the distance $h_c$ from c. It therefore lies on one of two parallels to c at the distance $h_c$ from c. (Locus 2, § 239.)

4. Point $C$ is such that $\angle ACB$ must equal $\angle C$. It therefore lies on the arc constructed on $AB$ as chord, the inscribed angles of which equal $\angle C$. (Locus 7, § 239.)




**Construction** is made so as to obtain the loci mentioned in steps 3 and 4. The circle cuts the line $RS$ at two points $C_1$ and $C_2$. $\triangle AC_1B$ and $\triangle AC_2B$ each satisfy the given conditions.

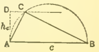
Proof and Discussion left to the pupil.

**Ex. 142.** Given the base and altitude of an isosceles triangle, construct the triangle.

**Ex. 143.** Construct an isosceles triangle having given the base and the radius of the circumscribed circle.

**Ex. 144.** Construct a rhombus having given its base and altitude.

**Ex. 145.** Construct a right triangle having given the hypotenuse and the length of the altitude upon it.

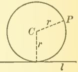



**Ex. 146.** Construct an isosceles triangle having given the base and the angle opposite the base.

**Ex. 147.** Construct a triangle having given the base, the altitude, and the radius of the circumscribed circle.

**Ex. 148.** Construct a triangle having given a side, an adjacent angle, and the radius of the circumscribed circle.

**Ex. 149.** Through a given point construct a circle with a given radius which shall be tangent to a given line.




**Analysis.** 1. Let the circle with the center $C$ pass through $P$ and be tangent to line $l$.

2. $C$ is $r$ distant from $P$. Hence it must lie on a circle having $P$ as center and $r$ as radius.

3. $C$ is $r$ distant from $l$. Hence $C$ must lie on one of two parallels to $l$ at the distance $r$ from $l$.

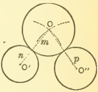
4. $C$ must be at the intersection of these two loci.

Construction, Proof, and Discussion left to the pupil.

**Ex. 150.** Construct a circle with a given radius which shall be tangent to each of two intersecting lines.

**Ex. 151.** Construct a circle which shall be tangent to each of two intersecting lines, tangent to one of them at a given point.

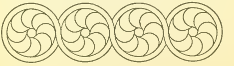
**Ex. 152.** Construct a circle through a given point not in a given line which shall be tangent to the given line at a given point in the line.




**Ex. 153.** Construct a circle having a given radius which shall be tangent to each of two given circles.

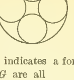
**Ex. 154.** A circular cylinder head 12 in. in diameter is to have holes bored in it for 12 1-in. bolts, equally spaced around the edge, with their centers $1\frac{1}{2}$ in. from the edge. Make a scale drawing of the cylinder head ($\frac{1}{4}'' = 1''$) and mark the centers for the 12 bolt holes.

**Ex. 155.** Determine how to construct the unit which is repeated in the design below. Construct it in a circle of 3-in. radius.

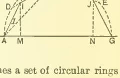


**Note.** — Supplementary Exercises 50 to 63, p. 288, can be studied now.

**Ex. 156.** Determine how to construct the left-hand figure. Notice that it is the basis for the artistic design at the right. Construct the left-hand figure in a circle of 4-in. diameter.



**Ex. 157.** The adjoining figure indicates a form of mansard roof. The chords $AB$, $CD$, $CE$, and $FG$ are all equal.

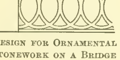



Construct such a roof outline for a building in which $AG$ is 30 ft. and the distance $IM$ is 10 ft. (Let 1 in. = 5 ft.)

Is $AI = IC = CJ = JG$?

Is the line $IJ \parallel AG$?

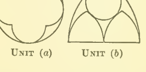
**Ex. 158.** Construct between two parallel lines a set of circular rings like those in the design below.



Design for Ornamental Stonework on a Bridge

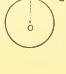
**Ex. 159.** The adjoining design is a panel for ornamental ironwork on a bridge.

Determine how to construct the fundamental units of the design, units (*a*) and (*b*).



Unit (*a*) Unit (*b*)

**Ex. 160.** Construct a tangent to a circle which will be parallel to a given line.



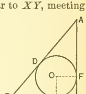

*Suggestion.* — Make an analysis based on the adjoining figure.

## Miscellaneous Exercises

**Ex. 161.** $AOB$ is a diameter of $\odot O$. $C$ is any point of $\widehat{AB}$. $D$ is the mid-point of $\widehat{BC}$ and $E$ is the mid-point of $\widehat{AC}$. Prove $\angle DOE$ is a right angle.

*Suggestion.* — Draw $CO$.

**Ex. 162.** Points $A$ and $B$ are on the diameter $XY$ of circle $O$ at equal distances from $O$. $CA$ and $DB$ are perpendicular to $XY$, meeting the semicircle at $C$ and $D$ respectively. Prove $ABDC$ is a rectangle.



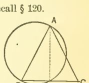

**Ex. 163.** If a circle is inscribed in a right triangle, the sum of its diameter and the hypotenuse is equal to the sum of the legs of the triangle.

**Ex. 164.** If $AB$ is a common external tangent of two circles which touch each other externally at $C$, prove $\angle ACB$ is a right angle.

*Suggestion.* — Draw the common tangent of the Ⓢ at $C$, meeting $AB$ at $D$.

**Ex. 165.** Prove that the bisector of the angle between two tangents to a circle passes through the center of the circle.

*Suggestions.* — Draw radii to the points of contact. Recall § 120.




**Ex. 166.** The circle drawn on one of the equal sides of an isosceles triangle as diameter bisects the base.

**Ex. 167.** Two circles are tangent externally at $C$. In one circle $\triangle ABC$ is inscribed, having one vertex at the point of contact of the circles. $AC$ and $BC$ are extended through $C$, meeting the other circle at $D$ and $E$ respectively. Prove $DE \parallel AB$.

*Suggestion.* — Draw the common tangent through point $C$.

**Ex. 168.** If a straight line be drawn through the point of contact of two circles which are tangent externally, terminating in their circumferences, the tangents at its extremities are parallel.

*Suggestion.* — Draw the common internal tangent of the circles.

**Ex. 169.** If $AB$ and $AC$ are the tangents from point $A$ to the circle $O$, $\angle BAC = 2\angle OBC$.

*Suggestions.* — 1. Draw $OA$. What relation does it bear to $BC$?
2. Compare $\angle BAO$ with $\angle OBC$.

**Ex. 170.** Euclid's construction for the tangent to a circle with center $M$ from a point $A$ outside of it is as follows:

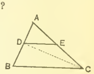
1. Draw the circle with center $M$ and radius $MA$.
2. Draw $MA$ intersecting the given ⊙ at $B$.
3. Draw $BC \perp MA$ at $B$, meeting the larger ⊙ at $C$.
4. Draw $MC$, intersecting the given ⊙ at $D$.

*Statement.* $AD$ is tangent to the given ⊙.
Make the construction and give the proof.

**Ex. 171.** Given a side and the diagonals of a parallelogram, construct the parallelogram.

**Ex. 172.** Through a given point within a circle, construct a chord equal to a given chord.
Is there any restriction on the location of the point?

**Ex. 173.** Construct a parallel to the side $BC$ of $\triangle ABC$ meeting $AB$ and $AC$ at $D$ and $E$ respectively, so that $DE$ will equal $EC$.



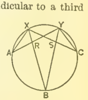

**Ex. 174.** If point $B$ bisects arc $AC$ of a circle, then $\angle A$ of $\triangle ABC$ equals $\angle C$.

**Ex. 175.** Prove that the bisectors of the angles of a circumscribed quadrilateral pass through a common point.

**Ex. 176.** Prove that two chords which are perpendicular to a third chord at its extremity are equal.

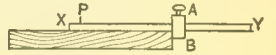
**Ex. 177.** If $\overset{\frown}{XA} = \overset{\frown}{YC}$ and $\overset{\frown}{BC} = \overset{\frown}{AB}$, prove $\triangle AXR \cong \triangle YCS$.

**Ex. 178.** In the figure for Ex. 177 draw $AC$ cutting $XB$ at $M$ and $YB$ at $N$. Prove $\triangle AXM \cong \triangle YCN$.




**Ex. 179.** A carpenter has a tool called a gauge which illustrates and applies one of the fundamental loci theorems.

The shaded rectangle represents the end of a board; the tool is upon the right-hand side of the board. $P$ is a marking point which extends to the under side of the tool. $AB$ is a movable part which can be fixed at any short distance from $P$ by means of a screw at $A$. By moving the gauge so that $AB$ is constantly against the edge of the board, the point $P$ traces upon the upper side of the board a line parallel to the edge of the board. Why is this so?



# BOOK III

## PROPORTION — SIMILAR POLYGONS

**242.** The **Ratio** of one number to another is the quotient of the first divided by the second.

Thus, the ratio of $a$ to $b$ is $\frac{a}{b}$; it is also written $a : b$.

The numerator is called the **Antecedent** and the denominator is called the **Consequent.**

Since a ratio is a fraction, it is subject to the usual rules for operations with fractions.

**243.** The ratio of two *concrete quantities* of the same kind is the ratio of their measures in terms of a common unit. (§ 212.)

Thus, the ratio of 350 lb. to 2 tons is $\frac{350}{4000}$ or $\frac{7}{80}$.

**Ex. 1.** Express the following ratios in their simplest form.

(*a*) 3 to 9. (*c*) $5x$ to $2x$. (*e*) $\frac{5}{8}$ to $\frac{3}{16}$. (*g*) 25 to 375.
(*b*) 12 to 2. (*d*) $6a^2$ to $15a^3$. (*f*) $\frac{2}{15}$ to $\frac{1}{3}$. (*h*) $a^2 - b^2$ to $a^3 - b^3$.

**Ex. 2.** A line 15 in. long is divided into two parts which have the ratio 2 : 3. Find the parts.

*Suggestion.* — If the short part contains $x$ in. and the long part $(15 - x)$ in., then $\frac{x}{15 - x} = \frac{2}{3}$. Complete the solution.

**Ex. 3.** Divide a line 63 in. long into two parts whose ratio is 3 : 4.

**Ex. 4.** Divide 36 into two parts such that the ratio of the greater diminished by 4 to the less increased by 3 will be 3 : 2.

**Ex. 5.** The ratio of the height of a tree to the length of its shadow on the ground is 17 : 20. Find the height of the tree if the length of the shadow is 110 feet.

**Ex. 6.** What is the ratio of: (*a*) a right angle to a straight angle? (*b*) a right angle to the perigon? (*c*) one angle of an equilateral triangle to the sum of all the angles of the triangle? (*d*) one side of a square to the perimeter of the square?

**244.** A **Proportion** is a statement that two ratios are equal; as, $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

This proportion is read "$a$ is to $b$ as $c$ is to $d$."

Thus, 1, 3, 5, and 15 form a proportion since $\frac{1}{3} = \frac{5}{15}$.

This means that 1 bears to 3 the same relation that 5 bears to 15.

The first and fourth terms of a proportion are called the **Extremes**, and the second and third terms, the **Means**.

In the proportion $a : b = c : d$, $a$ and $d$ are the extremes and $b$ and $c$ are the means; $a$ and $c$ are the antecedents, and $b$ and $d$ are the consequents.

**Ex. 7.** Select four numbers which form a proportion like the arithmetical illustration in § 244.

**Ex. 8.** (*a*) Is $\frac{3}{4} = \frac{5}{6}$? (*b*) Is $\frac{2}{4} = \frac{6}{8}$? (c) Is $\frac{3}{9} = \frac{4}{16}$?

**Ex. 9.** Find the value of the literal number in each of the following proportions.

(*a*) $\frac{x}{4} = \frac{5}{8}$. (c) $\frac{6}{16} = \frac{c}{8}$. (*e*) $\frac{2+x}{2} = \frac{5}{3}$.

(*b*) $\frac{10}{y} = \frac{2}{3}$. (*d*) $\frac{1}{8} = \frac{3}{z}$. (*f*) $\frac{3+t}{4-t} = \frac{5}{2}$.

**Ex. 10.** Find the value of $x$ in each of the following proportions.

(*a*) $\frac{a}{b} = \frac{c}{x}$. (*b*) $\frac{a}{3b} = \frac{x}{2c}$. (c) $\frac{r^2}{s} = \frac{rx}{t}$. (*d*) $\frac{mn}{p} = \frac{cn}{x}$.

**245.** Proportion is used in a great variety of ways.

EXAMPLE. — The cost of a number of articles of a given kind is "proportional" to the number of articles.

Thus, the cost of seven books is to the cost of three books of the same kind as 7 is to 3. Hence, if 3 books cost \$1.35, the cost of 7 books may be determined from the proportion $\frac{x}{1.35} = \frac{7}{3}$.

**Ex. 11.** Determine by proportion the cost of 13 yd. of cloth if the cost of 5 yd. of the same cloth is 70 ¢.

**Ex. 12.** Determine by proportion the distance an automobile will travel in one hour if it travels 2 mi. in 5 minutes.

**Ex. 13.** If a girl makes \$14.25 profit from 15 hens in one year, what profit can she expect from 50 hens, assuming the same average profit per hen?

**246.** The **Fourth Proportional** to three numbers $a$, $b$, and $c$ is the number $x$ in the proportion $a : b = c : x$.

Thus, the fourth proportional to 2, 3, and 4 is the number $x$ in $\frac{2}{3} = \frac{4}{x}$.

$$\therefore 2x = 12, \text{ or } x = 6.$$

**Note.** — The numbers *must* be placed in the proportion *in the order in which they are given* as in the illustrative example.

**Ex. 14.** Find the fourth proportional to:

(*a*) 2, 3, and 4.
(*b*) 3, 2, and 4.
(*c*) 4, 3, and 2.
(*d*) 25, 15, and 10.
(*e*) $3a$, $2b$, and $c$.
(*f*) $r$, $rs$, and $s$.

**247.** The **Third Proportional** to two numbers $a$ and $b$ is the number $x$ in the proportion $a : b = b : x$.

Thus, the third proportional to 2 and 3 is $x$ in $\frac{2}{3} = \frac{3}{x}$.

$$\therefore 2x = 9, \text{ or } x = \frac{9}{2}.$$

**Ex. 15.** Find the third proportional to:

(*a*) 3 and 5. (*b*) 2 and $5t$. (*c*) 5 and 10.

**248.** A **Mean Proportional** between two numbers $a$ and $b$ is the number $x$ in the proportion $a : x = x : b$.

Example. — A mean proportional between 2 and 3 is $x$ in:

$$\frac{2}{x} = \frac{x}{3}.$$

$$\therefore x^2 = 6, \text{ or } x = \pm\sqrt{6}.$$

There are two mean proportionals between any two numbers. The positive one is implied when "the" mean proportional is specified.

**Ex. 16.** Find the mean proportional between:

(*a*) 75 and 12.
(*b*) $3a$ and $a$.
(*c*) $2r^3t$ and $18rt$.
(*d*) $6\frac{2}{3}$ and $\frac{3}{5}$.

## FUNDAMENTAL THEOREMS OF PROPORTION

**249.** *The mean proportional between two numbers is the square root of their product.*

For the mean proportional between $a$ and $b$ is $x$ in:

$$\frac{a}{x}=\frac{x}{b}.$$

$$\therefore x^2 = ab; \text{ or } x=\sqrt{ab}.$$

**250.** *In a proportion, the product of the extremes is equal to the product of the means.*

If $\frac{a}{b}=\frac{c}{d}$, then $ad = bc$.

*Suggestion.* — Multiply both members of the proportion by $bd$.

Example. — Since $\frac{2}{3}=\frac{6}{9}$, $2 \times 9$ should equal $3 \times 6$. Does it?

**251.** *If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms also are equal.*

If $\frac{a}{b}=\frac{c}{x}$ and $\frac{a}{b}=\frac{c}{y}$, then $x = y$.

*Suggestion.* — Determine $x$ from the first proportion and $y$ from the second.

Example. — If $\frac{2}{3}=\frac{4}{x}$ and $\frac{2}{3}=\frac{4}{y}$, then $x$ should equal $y$.

**252.** *If the product of two numbers is equal to the product of two other numbers, one pair may be made the means of a proportion having the other pair as the extremes.*

If $mn = xy$, then $\frac{m}{x}=\frac{y}{n}$.

**Proof.** Dividing both members of the given equation by $xn$,

$$\frac{mn}{xn}=\frac{xy}{xn}, \text{ or } \frac{m}{x}=\frac{y}{n}.$$

Example. — Since $3 \times 8 = 6 \times 4$, $\frac{3}{6}$ should equal $\frac{4}{8}$. Does it?

**Ex. 17.** If $mn = xy$, prove:

(*a*) $\frac{m}{y}=\frac{x}{n}$. (Divide by $yn$.) (*b*) $\frac{x}{m}=\frac{n}{y}$. (*c*) $\frac{n}{x}=\frac{y}{m}$.

**Ex. 18.** Since $4 \times 5 = 2 \times 10$, write four proportions which involve 4, 5, 2, and 10.

**253.** *In any proportion, the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second is to the fourth.*

If $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{c}=\frac{b}{d}.$$

**Proof.** 1. Since $\frac{a}{b}=\frac{c}{d}$, then $ad = bc$. § 250

2. Since $ad = bc$, then $\frac{a}{c}=\frac{b}{d}$. § 252

EXAMPLE. — Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{2}{4}$ should equal $\frac{6}{12}$. Does it?

**Ex. 19.** Write each of the following proportions by alternation:

$$(a)\ \frac{2}{3}=\frac{10}{15};\quad (b)\ \frac{3}{5}=\frac{n}{r};\quad (c)\ \frac{r}{s}=\frac{x}{y}.$$

**254.** *In any proportion, the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth is to the third.*

If $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{b}{a}=\frac{d}{c}.$$

**Proof.** 1. Since $\frac{a}{b}=\frac{c}{d}$, then $ad = bc$. Why?

2. Since $ad = bc$, then $\frac{b}{a}=\frac{d}{c}$. Why?

EXAMPLE. — Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{6}{2}$ should equal $\frac{12}{4}$. Does it?

**Ex. 20.** Write each of the three proportions of Ex. 19 by inversion.

**Ex. 21.** Write the proportion $\frac{a}{b}=\frac{x}{y}$, by inversion, and then write the resulting proportion by alternation.

**255.** *In any proportion, the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the second, as the sum of the last two terms is to the fourth.*

If $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a+b}{b}=\frac{c+d}{d}.$$

**Proof.** 1. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}+1=\frac{c}{d}+1$. Why?

2. $\therefore \frac{a+b}{b}=\frac{c+d}{d}$. Algebraic addition.

EXAMPLE. Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{2+6}{6}$ should equal $\frac{4+12}{12}$. Does it?

**Ex. 22.** Write the three proportions of Ex. 19 by composition.

**Ex. 23.** Write the proportion $\frac{a}{b}=\frac{x}{y}$:

(*a*) By composition and the result by inversion.
(*b*) By inversion and the result by composition.
(*c*) By composition and the result by alternation.
(*d*) By alternation and the result by composition.

**256.** *In any proportion, the terms are in proportion by* **Division**; *that is, the first term minus the second is to the second, as the third minus the fourth is to the fourth.*

If $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a-b}{b}=\frac{c-d}{d}.$$

**Proof.** 1. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}-1=\frac{c}{d}-1$. Why?

(Complete the proof.)

**Ex. 24.** Write the three proportions of Ex. 19 by division.

**Ex. 25.** Write the proportion $\frac{a}{b}=\frac{x}{y}$:

(*a*) By division and the result by alternation.
(*b*) By alternation and the result by division.
(*c*) By inversion and the result by division.
(*d*) By division and the result by inversion.

**257.** *In any proportion, the terms are in proportion by* **Composition** *and* **Division**; *that is, the first term plus the second is to the first term minus the second, as the third term plus the fourth is to the third term minus the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

**Proof.** 1. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$. § 255

2. Also $\frac{a-b}{b}=\frac{c-d}{d}$. § 256

3. $\therefore \frac{a+b}{b} \div \frac{a-b}{b}=\frac{c+d}{d} \div \frac{c-d}{d}$. Why?

4. $\therefore \frac{a+b}{b} \cdot \frac{b}{a-b}=\frac{c+d}{d} \cdot \frac{d}{c-d}$,

or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Example. — Since $\frac{10}{2}=\frac{15}{3}$, then $\frac{10+2}{10-2}=\frac{15+3}{15-3}$. Does it?

**Ex. 26.** Write the proportion $\frac{a}{b}=\frac{x}{y}$:

(*a*) By composition and division and the result by inversion.
(*b*) By inversion and the result by composition and division.
(*c*) By composition and division and the result by alternation.
(*d*) By alternation and the result by composition and division.

**258.** *In any proportion, like powers or like roots of the terms are in proportion.*

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, and also $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

The first of these conclusions follows from raising both members of the given proportion to the $n$th power, and the second from taking the $n$th roots of both members.

**Ex. 27.** If $\frac{a}{b}=\frac{c}{d}$, prove: (*a*) $\frac{ma}{b}=\frac{mc}{d}$; (*b*) $\frac{ma}{mb}=\frac{c}{d}$.

**259.** The preceding paragraphs give some of the essential facts about the subject of proportion. Remember that the terms in every case are *numbers.* We shall be dealing with ratios and proportions of *geometrical magnitudes.* However, as explained in § 243, we replace the magnitudes themselves by their measures in terms of common units so that the terms are again numbers. The consequence is that the theorems about proportions all apply to the proportions which we shall encounter.

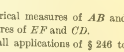
Thus, if $AB$, $CD$, $EF$, and $GH$ are four segments such that

$$\frac{AB}{CD}=\frac{EF}{GH}, \text{ then } AB \times GH = EF \times CD.$$

This means that the product of the numerical measures of $AB$ and $GH$ equals the product of the numerical measures of $EF$ and $CD$.

A similar interpretation must be given to all applications of § 246 to § 258 inclusive.

## PROPORTIONAL LINE-SEGMENTS




**260. Introduction.** If $AE = EB$ and $CF = FD$, then $\frac{AE}{EB}=\frac{CF}{FD}$ since each ratio equals 1. Again, in the same figure, if $G$ bisects $AE$ and $H$ bisects $CF$, then $\frac{AG}{GB}=\frac{1}{3}$ and also $\frac{CH}{HD}=\frac{1}{3}$; hence $\frac{AG}{GB}=\frac{CH}{HD}$. This means that $AG$ bears to $GB$ the same relation that $CH$ bears to $HD$. $G$ and $H$ are said to divide $AB$ and $CD$ proportionally.

**Def.** Two **line-segments** are **divided proportionally** when the segments of one have the same ratio as the corresponding segments of the other.

**Ex. 28.** Draw a $\triangle ABC$, having $AB = 2$ in., $AC = 3$ in., and $BC = 4$ in. Place $X$ on $AB$, so that $AX = .5$ in. Draw from $X$ a parallel to $BC$ meeting $AC$ at $Y$.

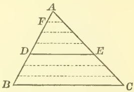
(*a*) Measure $AY$ and $YC$, and determine the ratio of $AY$ to $YC$.

(*b*) What is the ratio of $AX$ to $XB$?

(*c*) Do the sides appear to be divided proportionally?

## Proposition I. Theorem

**261.** *A parallel to one side of a triangle, intersecting the other two sides, divides the other two sides proportionally.*



**Hypothesis.** In $\triangle ABC$, $DE \parallel BC$, meeting $AB$ at $D$ and $AC$ at $E$.

**Conclusion.** $\frac{AD}{DB} = \frac{AE}{EC}$.

Case I. Suppose that $AD$ and $DB$ are commensurable. § 211

**Proof.** 1. Let $AF$ be a common measure contained 4 times in $AD$ and 3 times in $DB$.

$$\therefore \frac{AD}{DB} = \frac{4}{3}.$$

2. Draw ∥s to $BC$ through the points of division on $AB$. Then $AC$ will be divided into equal segments, of which 4 are in $AE$ and 3 are in $EC$. § 147

3. $$\therefore \frac{AE}{EC} = \frac{4}{3}.$$

4. Then, from steps (1) and (3), $\frac{AD}{DB} = \frac{AE}{EC}$. Ax. 1, § 51

Case II. Suppose that segments $AD$ and $DB$ are incommensurable. § 211

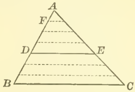
The proof given for Case I will not apply, as no common measure with which to divide both $AD$ and $DB$ can be found.

*The theorem is true however for the incommensurable case also.* The proof is given in § 424 and, if desired, may be read at this time.

**262. Cor.** 1. Since $AD : DB = AE : EC$, then

$$\frac{AD+DB}{DB}=\frac{AE+EC}{EC}, \text{ or } \frac{AB}{DB}=\frac{AC}{EC}. \qquad \text{Why?}$$

That is, one side is to its lower segment as the other side is to its lower segment.



**263. Cor.** 2. Since $\frac{AD}{DB}=\frac{AE}{EC}$, then $\frac{DB}{AD}=\frac{EC}{AE}$. Why?

$$\therefore \frac{DB+AD}{AD}=\frac{EC+AE}{AE}, \text{ or } \frac{AB}{AD}=\frac{AC}{AE}. \qquad \text{Why?}$$

That is, one side is to its upper segment as the other side is to its upper segment.

**264.** Numerous other proportions may be derived from the proportions obtained in §§ 261–263 by making allowable changes in them.

| | | |
|---|---|---|
| From § 261, | $AD : AE = DB : EC.$ | Why? |
| From § 262, | $BD : AB = EC : AC.$ | Why? |
| From § 263, | $AD : AB = AE : AC.$ | Why? |

**Note.** — In every case, corresponding segments occur in the proportion in the same manner.

**265.** For convenience, reference may be made to any of the proportions developed in §§ 261–264 by quoting the authority:

*A parallel to one side of a triangle divides the other two sides proportionally.*

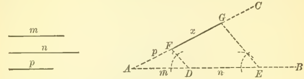
**Ex. 29.** If, in the figure of § 261, $AD$ is $\frac{1}{3}$ of $BD$, what is the ratio of $AE$ to $EC$?

**Ex. 30.** If $AD = 8$ in., $DB = 5$ in., and $EC = 6$ in., find $AE$.

**Ex. 31.** If $AB = 12$ in., $AC = 15$ in., and $AE = 6$ in., find $AD$.

## Proposition II. Problem

**266.** *Construct the fourth proportional to three given segments.*




**Given** segments $m$, $n$, and $p$.

**Required** to construct the fourth proportional to $m$, $n$, and $p$.

**Analysis.** 1. Let $x$ represent the fourth proportional.

Then $m : n = p : x$.

2. This suggests the following construction.

**Construction.** 1. On side $AB$ of a convenient angle, $\angle BAC$, take $AD = m$, and $DE = n$; on $AC$, take $AF = p$.

2. Draw $DF$ and construct $EG \parallel DF$, meeting $AC$ at $G$.

**Statement.** Then $FG$ is the fourth proportional to $m$, $n$, and $p$.

[Proof to be given by the pupil.]

**267. Cor.** *Construct the third proportional to m and n.*

**Analysis.** 1. Let $x$ represent the third proportional to $m$ and $n$.

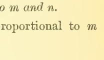
2. Then $m : n = n : x$.

[Construction and proof to be given by the pupil.]

**Ex. 32.** Construct the fourth proportional to segments which are 2 in., 1 in., and 3 in. in length. Measure the resulting segment. Verify your work by computing the fourth proportional to 2, 1, and 3, as in Ex. 14.

**Ex. 33.** Let $OB$ be any line within $\angle AOC$ and $Y$ and $Y'$ any two points on $OB$. Let $YX$ and $Y'X'$ be perpendiculars to $OA$, and $YZ$ and $Y'Z'$ be perpendiculars to $OC$. Prove that $OX : OX' = OZ : OZ'$.

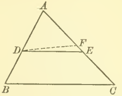
**Ex. 34.** A line drawn parallel to the bases of a trapezoid and intersecting the non-parallel sides, divides the non-parallel sides proportionally.




Prove $BE : EA = CF : FD$.

## Proposition III. Theorem

**268.** *A line which divides two sides of a triangle proportionally is parallel to the third side.*




**Hypothesis.** In $\triangle ABC$, $DE$ intersects $AB$ and $AC$ so that

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

**Conclusion.** $DE \parallel BC$.

**Proof.** 1. Assume $DF \parallel BC$, meeting $AC$ at $F$.

2. Then $\frac{AB}{AD}=\frac{AC}{AF}$. § 265

3. But $\frac{AB}{AD}=\frac{AC}{AE}$. Why?

4. $\therefore AF=AE$. § 251

5. $\therefore F$ coincides with $E$, and $DE$ with $DF$. Why?

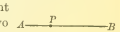
6. $\therefore DE \parallel BC$. Step 1

**Ex. 35.** If $AD=3$ in., $AB=12$ in., $AC=10$ in., and $AE=2.5$ in., is $DE \parallel BC$?

**Ex. 36.** If $AD=5$ in., $BD=10$ in., $AE=6$ in., and $EC=11$ in., is $DE \parallel BC$?

**Ex. 37.** What relation is there between Prop. I and Prop. III?

**269. Def.** If $P$ is a point of segment $AB$, then $P$ **divides** $AB$ **internally** into two segments $AP$ and $PB$.

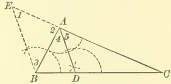



**Ex. 38.** Construct a $\triangle ABC$ having $AB=2$ in., $BC=4$ in., and $AC=4.5$ in. Let the bisector of $\angle B$ meet $AC$ at $D$. Measure $AD$ and $DC$. Compare the ratio of $AB$ to $BC$ with the ratio of $AD$ to $DC$.

**Note.** — Supplementary Exercises 1 to 3, p. 289, can be studied now.

## Proposition IV. Theorem

**270.** *In any triangle, the bisector of an interior angle divides the opposite side internally into segments proportional to the adjacent sides of the triangle.*




**Hypothesis.** $AD$ bisects $\angle A$ of $\triangle ABC$, meeting $BC$ at $D$.

**Conclusion.** $$\frac{BD}{DC} = \frac{BA}{AC}.$$

**Proof.** 1. Draw $BE \parallel DA$, meeting $CA$ extended at $E$.

2. Then, in $\triangle EBC$, $\dfrac{BD}{DC} = \dfrac{EA}{AC}$. Why?

Prove now that $BA = EA$ and substitute it for $EA$ in step 2.

*Suggestions.* — (1) Recall § 123. (2) Compare $\angle 1$ and $\angle 3$ with $\angle 5$ and $\angle 4$ respectively, and use the hypothesis.

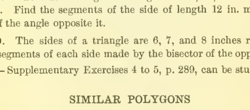
**Ex. 39.** The sides of a given triangle are 10, 20, and 12 inches respectively. Find the segments of the side of length 12 in. made by the bisector of the angle opposite it.

**Ex. 40.** The sides of a triangle are 6, 7, and 8 inches respectively. Find the segments of each side made by the bisector of the opposite angle.

**Note.** — Supplementary Exercises 4 to 5, p. 289, can be studied now.

### SIMILAR POLYGONS

**271. Introduction.** The triangles below are similar triangles. Notice that they appear to have the same shape.



**Ex. 41.** Construct a $\triangle ABC$ having $AB = 2$ in., $\angle A = 50°$, and $\angle B = 80°$. Also construct a $\triangle A'B'C'$, having $A'B' = 4$ in., $\angle A' = 50°$, and $\angle B' = 80°$.

(*a*) Do the triangles appear to be similar, in the sense that they have the same shape?

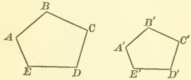
(*b*) Determine the lengths of $BC$, $B'C'$, $AC$, and $A'C'$. Then determine the approximate values of the ratios: $AB : A'B'$; $BC : B'C'$; $AC : A'C'$.

(*c*) Do the ratios appear to be about equal?

**272. Def.** Two **polygons** are **similar** ($\sim$) if:

(1) Their homologous angles are equal;

(2) Their homologous sides are proportional.




Thus, $ABCDE \sim A'B'C'D'E'$ if:

(1) $\angle A = \angle A'$; $\angle B = \angle B'$; $\angle C = \angle C'$; etc. and

(2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \cdots$ etc.

The ratio of any two homologous sides of two similar polygons is called the **Ratio of Similitude** of the polygons.

**Note.** — Two polygons may have their homologous angles equal and still fail to be similar; as a square and a rectangle.

**Ex. 42.** Are two squares similar? Why?

**Ex. 43.** Are two equilateral triangles similar? Why?

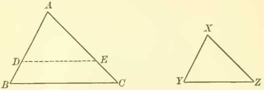
**Ex. 44.** Are two rectangles necessarily similar?

**Ex. 45.** The sides of one triangle are 1 in., 1.5 in., and 2 in. respectively. The shortest side of a similar triangle is 2 in. What are the other sides of the second triangle? Construct the two triangles.

**Ex. 46.** The sides of one pentagon are 3, 4, 5, 8, and 11 in. respectively. The shortest side of a similar pentagon is 9 in. How long are the remaining sides of the second pentagon?

## Proposition V. Theorem

**273.** *Two triangles are similar if they are mutually equiangular.*



**Hypothesis.** In $\triangle ABC$ and $\triangle XYZ$:

$$\angle A = \angle X;\ \angle B = \angle Y;\ \angle C = \angle Z.$$

**Conclusion.** $\triangle ABC \sim \triangle XYZ.$

**Plan.** We must prove the homologous sides proportional.

**Proof.** 1. Place $\triangle XYZ$ in the position $ADE$, $\angle X$ coinciding with $\angle A$, and vertices $Y$ and $Z$ falling at $D$ and $E$ respectively.

2. Since $\angle ADE = \angle B$, then $DE \parallel BC$. Why?

3. $\therefore \dfrac{AB}{AD} = \dfrac{AC}{AE}$. Why?

4. That is $\dfrac{AB}{XY} = \dfrac{AC}{XZ}$.

5. By placing $\triangle XZY$ so that $\angle Y$ coincides with its equal $\angle B$, it may be proved that

$$\frac{AB}{XY} = \frac{BC}{YZ}.$$

6. From steps (4) and (5), $\dfrac{AB}{XY} = \dfrac{AC}{XZ} = \dfrac{BC}{YZ}$. Why?

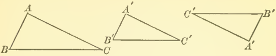
7. $\therefore \triangle ABC \sim \triangle XYZ.$ § 272

**274. Cor. 1.** *Two triangles are similar if two angles of one are equal respectively to two angles of the other.*

*Suggestion.* — Recall § 111.

**275. Cor. 2.** *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

**276. Cor.** 3. *Two triangles are similar if their sides are parallel each to each.*

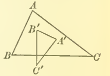



**Hypothesis.** In $\triangle ABC$ and (either) $\triangle A'B'C'$:
$AB \parallel A'B'$; $BC \parallel B'C'$; $AC \parallel A'C'$.

**Conclusion.** $\triangle ABC \sim \triangle A'B'C'$.

*Suggestion.* — Recall § 105.

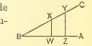
**277. Cor.** 4. *Two triangles are similar if their sides are perpendicular each to each.*




**Hypothesis.** In $\triangle ABC$ and $\triangle A'B'C'$;
$AB \perp A'B'$; $BC \perp B'C'$; $AC \perp A'C'$.

**Conclusion.** $\triangle ABC \sim \triangle A'B'C'$.

**Ex. 47.** Construct any triangle $ABC$. Upon a segment $XY$ which equals 2 $AB$ construct a triangle similar to $\triangle ABC$. (§ 274)




**Ex. 48.** If $X$ and $Y$ are any two points on the side $BC$ of acute angle $ABC$ and $XW$ and $YZ$ are perpendiculars to $AB$, then $\triangle BXW \sim \triangle BYZ$.

**Ex. 49.** If chords $AB$ and $CD$ of a circle intersect at $E$ within the circle, then $\triangle AED \sim \triangle BEC$.

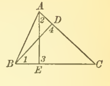
**Ex. 50.** If $AD$ and $CE$ are the altitudes drawn from $A$ and $C$ respectively in $\triangle ABC$, then $\triangle ABD \sim \triangle CBE$.

**Ex. 51.** In the figure for Ex. 50, if $AD$ intersects $CE$ at $O$, prove $\triangle AEO$ is similar to $\triangle CDO$.

**Ex. 52.** If a line be drawn parallel to the base of a triangle intersecting the other two sides, the triangle formed is similar to the given triangle.

**278. Homologous sides of similar triangles are proportional.** Homologous sides lie opposite equal angles. The following exercise illustrates a *device for selecting the homologous sides* and for forming the three equal ratios.

ILLUSTRATIVE EXERCISE




**Hypothesis.** In $\triangle ABC$:
$AE \perp BC$;
$BD \perp AC$.

**Conclusion.** $\frac{BD}{AE}=\frac{BC}{AC}=\frac{CD}{EC}$.

**Proof.** 1. In $\triangle AEC$ and $\triangle BDC$:
$\angle 3=\angle 4$; $\angle C\equiv\angle C$; Why?

2. $\therefore \triangle AEC \sim \triangle BDC$. Why?

3. In $\triangle AEC$ and $\triangle BDC$:

$$\begin{array}{c|c|c} AC & \angle 3=\angle 4 & BC \\ AE & \angle C=\angle C & BD \\ EC & \angle 2=\angle 1 & DC \end{array}$$

$\left\{\begin{array}{c}\text{Read}\\ \text{Note 1}\\ \text{now.}\end{array}\right\}$

4. $\therefore \frac{AC}{BC}=\frac{AE}{BD}=\frac{EC}{DC}$. § 278

**Note 1.** — (*a*) Below $\triangle AEC$, write its three angles and to the left of them write the sides which are opposite them in $\triangle AEC$.

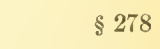
(*b*) Opposite the angles of $\triangle AEC$ write the equal angles of $\triangle BDC$.

(*c*) Beside the angles of $\triangle BDC$, write the sides opposite them in $\triangle BDC$.

(*d*) Then $AC$ and $BC$ are homologous sides; also $AE$ and $BD$; also $EC$ and $DC$.

**Note 2.** — Sometimes only two equal ratios are wanted. In that case, cross out in step 4 the ratio that is not wanted.

**279. Principle IV.** Four segments can be proved proportional by proving them homologous sides of similar triangles.




**Ex. 53.** If $AB \perp BC$ and $DC \perp BC$, prove that
$$\frac{AB}{DC}=\frac{BO}{CO}=\frac{AO}{DO}.$$

**Ex. 54.** If the altitudes $AD$ and $CE$ of $\triangle ABC$ intersect at $F$, prove that $AF : CF = EF : DF$.

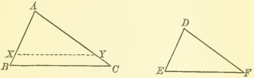
**Ex. 55.** In the figure for Ex. 54, prove that $AD : CE = AB : CB$.

**Ex. 56.** Prove that the diagonals of a trapezoid divide each other so that the corresponding segments are proportional.

**Note.** — Supplementary Exercises 6 to 13, p. 290, can be studied now.

## Proposition VI. Theorem

**280.** *Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.*



**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:

$$\angle A = \angle D;\ \frac{AB}{DE} = \frac{AC}{DF}.$$

**Conclusion.** $\triangle ABC \sim \triangle DEF.$

**Proof.** 1. Place $\triangle DEF$ in the position $AXY$, $\angle D$ coinciding with its equal $\angle A$, $E$ falling at $X$ and $F$ at $Y$.

| | | |
|---|---|---|
| 2. Then | $\frac{AB}{AX} = \frac{AC}{AY}.$ | Hyp. and step 1 |
| 3. | $\therefore XY \parallel BC.$ | Why? |
| 4. | $\therefore \triangle AXY \sim \triangle ABC.$ | Give the full proof. |
| 5. | $\therefore \triangle DEF \sim \triangle ABC.$ | Why? |

**Ex. 57.** Two segments $AOB$ and $COD$ intersect at $O$ so that $AO = 3\ OB$ and $CO = 3\ OD$. Prove $AC = 3\ BD$.

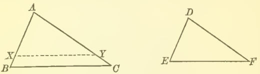
**Ex. 58.** $\angle A$ of $\triangle ABC$ is a right angle. From $E$, any point of $AC$, $ED$ is drawn perpendicular to $BC$, meeting it at $D$. (*a*) Examine the figure to discover a pair of similar triangles; (*b*) prove the triangles similar; (*c*) from these triangles determine the three equal ratios of sides of the triangle.

**Ex. 59.** $\angle ABC$ is an acute angle. $CD$ is perpendicular to $AB$ and $AF$ is perpendicular to $BC$. (*a*) Discover a pair of similar triangles; (*b*) prove the triangles similar; (*c*) write down three equal ratios of sides of these triangles.

**Ex. 60.** The shadow of a chimney is 36 yd. long. At the same time the shadow of a stake 2 yd. long is 1.5 yd. in length. How high is the chimney?

## Proposition VII. Theorem

**281.** *Two triangles are similar if their homologous sides are proportional.*



**Hypothesis.** In $\triangle ABC$ and $\triangle DEF$:

$$\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}.$$

**Conclusion.** $\triangle ABC \sim \triangle DEF.$

**Proof.** 1. On $AB$, take $AX = DE$; on $AC$ take $AY = DF$. Draw $XY$.

| | | |
|---|---|---|
| 2. Then | $\dfrac{AB}{AX}=\dfrac{AC}{AY}.$ | By hyp. and step 1 |
| 3. | $\therefore XY \parallel BC.$ | Why? |
| 4. | $\therefore \triangle AXY \sim \triangle ABC.$ | Give the full proof. |
| 5. | $\therefore \dfrac{AB}{AX}=\dfrac{BC}{XY}$, or $\dfrac{AB}{DE}=\dfrac{BC}{XY}.$ | Why? |
| 6. But | $\dfrac{AB}{DE}=\dfrac{BC}{EF}.$ | Why? |
| 7. | $\therefore XY = EF.$ | § 251 |
| 8. Then | $\triangle AXY \cong \triangle DEF.$ | Give the full proof. |
| 9. | $\therefore \triangle DEF \sim \triangle ABC.$ | Why? |

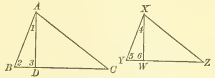
Note. — Notice that $\triangle DEF$ is not superposed on $\triangle ABC$; that, rather, $\triangle AXY$ is constructed, and is proved similar to $\triangle ABC$ and congruent to $\triangle DEF$.

**Ex. 61.** Construct any scalene triangle. Then construct a triangle whose sides are double the corresponding sides of the first triangle. Are the two triangles similar?

**Ex. 62.** Determine three segments which shall bear to the sides of a given triangle the ratio 3 : 2. Then construct the triangle having the new segments as sides. Are the two triangles similar?

## Proposition VIII. Theorem

**282.** *Homologous altitudes of similar triangles have the same ratio as any two homologous sides.*




**Hypothesis.** $\triangle ABC \sim \triangle XYZ$.

$AD$ and $XW$ are homologous altitudes.

**Conclusion.** $\dfrac{AD}{XW} = \dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$.

**Proof.** 1. In rt. $\triangle ABD$ and rt. $\triangle XYW$:

$\angle 2 = \angle 5$. § 272

2. $\therefore \triangle ABD \sim \triangle XYW$. Why?

3. $\therefore \dfrac{AD}{XW} = \dfrac{AB}{XY}$. Why?

4. But $\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$. Why?

5. $\therefore \dfrac{AD}{XW} = \dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$. Ax. 1, § 51

**Note.** — It can be proved that any two homologous lines of similar triangles are proportional to any two homologous sides.

**Ex. 63.** The base and altitude of a triangle are 5 ft. and 3 ft. respectively. If the homologous base of a similar triangle is 7 ft., find its homologous altitude.

**Ex. 64.** Prove that the bisectors of homologous angles of similar triangles have the same ratio as any two homologous sides.

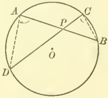
*Suggestion.* — The length of the bisector is the length of the segment of the bisector between the vertex of the angle and the opposite side of the triangle.

**Ex. 65.** Prove that two homologous medians of two similar triangles have the same ratio as any two homologous sides of the triangles.

*Suggestion.* — Use § 280.

## Proposition IX. Theorem

**283.** *If two chords are drawn through a fixed point within a circle, the product of the segments of one is equal to the product of the segments of the other.*




**Hypothesis.** $AB$ and $CD$ are any two chords of $\odot\ O$ intersecting at point $P$.

**Conclusion.** $AP \cdot PB = DP \cdot PC.$

**Analysis.** 1. If $AP \cdot PB = DP \cdot PC$, then $AP : PC = DP : PB$. § 252

2. $\therefore$ try to prove $\triangle APD \sim \triangle PBC$. § 279

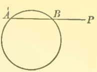
| | | |
|---|---|---|
| **Proof.** 1. | Draw $AD$ and $BC$. | |
| 2. | $\triangle APD \sim \triangle PBC.$ | Give the full proof. |
| 3. | $\therefore \dfrac{AP}{PC} = \dfrac{DP}{PB}.$ | Why? |
| 4. | $\therefore AP \cdot PB = PC \cdot DP.$ | Why? |

**284. Principle V.** To prove that the product of two segments equals the product of two other segments, first derive from the equation a proportion by § 252 and then try to proceed as in § 279.

This principle is illustrated in the proof of § 283.

**Ex. 66.** Two chords of a circle intersect so that the segments of one are 4 in. and 5 in. respectively. If the shorter segment of the other is 6 in., what is the longer segment?

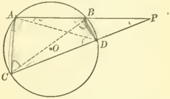
**Ex. 67.** In a circle whose diameter is 16 in., a chord 14 in. long is drawn through a point which is 4 in. from the center. What are the two segments of the chord? (Represent one segment by $x$.)




**285.** If a secant $PA$ is drawn to a circle from a point $P$, cutting the circle at point $B$, then $PA$ is called the **whole secant**, $PB$ the **external segment**, and $AB$ the **internal segment.**

## Proposition X. Theorem

**286.** *If any two secants are drawn through a fixed point outside a circle, the product of one and its external segment equals the product of the other and its external segment.*




**Hypothesis.** $ABP$ and $CDP$ are two secants of $\odot$ $O$.

**Conclusion.** $AP \cdot BP = CP \cdot DP.$

*Suggestion.* — Make an analysis and a proof similar to that for Proposition IX.

**Ex. 68.** From a point $P$, a secant 18 in. long is drawn to a circle; the external segment is 3 in. The external segment of a second secant from the same point is 6 in. long. How long is the whole secant?

**Ex. 69.** A secant is drawn from point $P$ to a circle. The external segment is 4 in. and the internal segment is 6 in. How long must a second secant be in order that its internal segment shall be 3 in.?

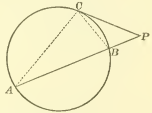
**Ex. 70.** If from an exterior point $P$, any number of secants be drawn, the product of the whole secant and the external segment is constant.

**Note.** — The conclusion means that the product of the whole secant and the external segment is the same for each secant.

**Ex. 71.** Prove that the product of the segments of one diagonal of an inscribed quadrilateral is equal to the product of the segments of the other diagonal.

## Proposition XI. Theorem

**287.** *If a secant and a tangent are drawn to a circle from the same point outside a circle, the square of the tangent is equal to the product of the whole secant and its external segment.*



**Hypothesis.** $PC$ is a tangent to $\odot\ O$; secant $PA$ intersects the $\odot$ at $B$ and $A$.

**Conclusion.** $\overline{CP}^2 = AP \cdot BP.$

(Analysis and proof left to the pupil.)

**Note.** — Proposition XI may be stated: *If a secant and a tangent are drawn to a circle from the same point outside the circle, the tangent is the mean proportional between the whole secant and its external segment.*

For, when $\overline{CP}^2 = AP \cdot BP$, $AP : CP = CP : BP$. Why?

**Ex. 72.** The length of the tangent to a circle from a point outside is 12 in. What must be the length of a secant from the same point in order that the external segment will be 8 in.?

**Ex. 73.** If altitudes $AD$ and $BE$ of $\triangle ABC$ intersect at $F$, prove that the product of the segments of one is equal to the product of the segments of the other. (Apply § 284.)

**Ex. 74.** If $AD$ and $BE$ are two altitudes of $\triangle ABC$, then $AD \cdot BC = BE \cdot AC$.

**Ex. 75.** If $\angle A$ of $\triangle ABC$ is a right angle and $ED$ is drawn perpendicular to $CB$ from any point $E$ of $AB$, meeting $CB$ at $D$, then $EB \cdot AB = CB \cdot DB$.

**Ex. 76.** If altitudes $AD$ and $BE$ of $\triangle ABC$ intersect at $F$, then:

(*a*) $BE \cdot EF = AE \cdot EC$; (*b*) $AD \cdot DF = BD \cdot DC$.

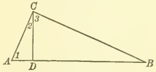
**Note.** — Supplementary Exercises 14 to 18, p. 290, can be studied now.

## Proposition XII. Theorem

**288.** *If the altitude be drawn to the hypotenuse of a right triangle:*

I. *The altitude is the mean proportional between the segments of the hypotenuse;*

II. *Each leg is the mean proportional between the whole hypotenuse and the adjacent segment.*



**Hypothesis.** In $\triangle ABC$, $\angle C$ is a rt. $\angle$.

$CD \perp AB$.

**Conclusion. I.** $\dfrac{AD}{CD} = \dfrac{CD}{DB}$.

**Proof.** 1. $\angle 3$ and $\angle 1$ are each complements of $\angle 2$. Why?

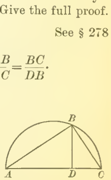
$\therefore \angle 1 = \angle 3$. Why?

2. $\therefore \triangle ADC \sim \triangle DCB$. Give the full proof.

3. $\therefore \dfrac{AD}{CD} = \dfrac{CD}{DB}$. See § 278

**Conclusion. II.** (*a*) $\dfrac{AB}{AC} = \dfrac{AC}{AD}$; (*b*) $\dfrac{AB}{BC} = \dfrac{BC}{DB}$.

**Plan.** For (*a*) prove $\triangle ABC \sim \triangle ACD$.



**289. Cor.** *If a perpendicular be drawn from any point on a circle to a diameter:*

(*a*) *The perpendicular is the mean proportional between the segments of the diameter.*

(*b*) *The chord joining the point to either extremity of the diameter is the mean proportional between the whole diameter and the segment of it adjacent to the chord.*

**Ex. 77.** Find the altitude drawn to the hypotenuse of a right triangle if it divides the hypotenuse into two segments whose lengths are 3 in. and 12 in. respectively. Find each leg of the right triangle.

**Ex. 78.** The hypotenuse of a right triangle is 20 in. and the perpendicular to it from the opposite vertex is 8 in. Find the segments of the hypotenuse, and the two legs of the triangle.

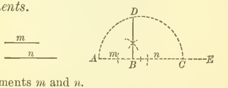
**Ex. 79.** $C$ and $D$ are respectively the mid-points of a chord $AB$ and its subtended arc. If $AD = 12$ and $CD = 8$, what is the diameter of the circle?

*Suggestion.* — $DC$ extended passes through the center of the circle.

**Ex. 80.** A chord of a circle is 20 in. in length. Its mid-point is 5 in. from the mid-point of its arc. Find the diameter of the circle.

## Proposition XIII. Theorem

**290.** *Construct the mean proportional between two given segments.*




**Given** segments $m$ and $n$.

**Required** to construct the mean proportional between $m$ and $n$.

**Construction.** 1. On line $AE$ take $AB = m$ and $BC = n$.

2. Construct semicircle $ADC$ on $AC$ as diameter.

3. Construct $DB \perp AC$, meeting the semicircle at $D$.

**Statement.** $DB$ is the mean proportional between $m$ and $n$.

(The proof is to be given by the pupil. See § 289.)

**Ex. 81.** Construct a segment equal to $a\sqrt{3}$ where $a$ is any segment whatever.

**Analysis.** 1. Let $x = a\sqrt{3}$. Then $x^2 = 3a^2$. Why?

2. $\therefore 3a : x = x : a$. § 252

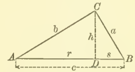
3. $\therefore x$ is the mean proportional between $a$ and $3a$.

(Construction is to be given by the pupil.)

**Ex. 82.** Construct a segment equal to $\sqrt{3ab}$ where $a$ and $b$ are any given segments.

## Proposition XIV. Theorem

**291.** *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*




**Hypothesis.** In $\triangle ABC$, $\angle C$ is a right angle.

**Conclusion.** $c^2 = a^2 + b^2$.

**Proof.** 1. Draw $CD \perp AB$. Let $AD = r$, and $DB = s$.

2. Then $\frac{c}{a} = \frac{a}{s}$ and $\frac{c}{b} = \frac{b}{r}$. § 288, II

3. $\therefore a^2 = cs$ and $b^2 = cr$. Why?

4. $\therefore a^2 + b^2 = cs + cr$
$= c(s + r)$.

5. $\therefore a^2 + b^2 = c \cdot c$, or $a^2 + b^2 = c^2$.

**Note.** — This theorem is called the Pythagorean Theorem, after Pythagoras, who formulated it. The theorem was evidently known even to the Egyptians. This proof of the theorem is attributed to Hindu mathematicians. In Book IV, we shall study Euclid's proof of the theorem — a strictly geometric proof, whereas this is more an algebraic one.

**292. Cor.** *The square of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.*

**Ex. 83.** How long must a rope be to run from the top of a 12-foot tent pole to a point 16 ft. from the foot of the pole?

**Ex. 84.** The diameters of two concentric circles are 14 in. and 50 in., respectively. Find the length of a chord of the greater circle which is tangent to the smaller.

**Ex. 85.** A baseball diamond is a square whose sides are each 90 ft. long. What is the distance from "first" to "third"?

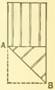
**Ex. 86.** Find the formula for the diagonal of a square whose side is $s$. By this formula determine the diagonal when: (*a*) $s = 10$; (*b*) $s = 15$.

**Ex. 87.** The equal sides of an isosceles trapezoid are each 10 in. long. One of the bases is 30 in., and the other is 42 in. in length. What is the altitude of the trapezoid?

**Ex. 88.** Find the length of the altitude of an equilateral triangle if each side is 10 in.

**Ex. 89.** Derive the formula for the length of the altitude of an equilateral triangle if each side of the triangle is $s$.

By this formula determine the altitude when: (*a*) $s = 6$ in.; (*b*) $s = 13$ in.




**Ex. 90.** A piece of silk 27 in. wide is folded "on the bias" along the line $AB$. How long is $AB$?

**Ex. 91.** Find the length of each side of a rhombus if the diagonals are 6 in. and 8 in. respectively.

**Ex. 92.** If $AD$ is the perpendicular from $A$ to $BC$ of $\triangle ABC$, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{DB}^2 - \overline{CD}^2.$$

**Plan.** Find an expression for $\overline{AB}^2$ and $\overline{AC}^2$; then subtract the latter from the former.

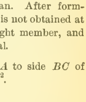
**Note.** — This might be called a "common sense" plan. After forming in this manner the left member, if the right member is not obtained at once, then proceed to form in the same manner the right member, and afterwards try to prove the two values obtained are equal.

**Ex. 93.** If $D$ is any point in the altitude from $A$ to side $BC$ of $\triangle ABC$, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{DB}^2 - \overline{DC}^2$.

*Suggestion.* — Read the note under Ex. 92.

**Ex. 94.** If a parallel to hypotenuse $AB$ of right triangle $ABC$ meets $AC$ and $BC$ at $D$ and $E$ respectively, prove that

$$\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2.$$



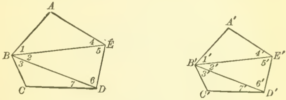

**Ex. 95.** If perpendiculars $PF$, $PD$, and $PE$ be drawn from any point $P$ within an acute-angled triangle $ABC$ to sides $AB$, $BC$, and $CA$ respectively, prove that

$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

**Note.** — Supplementary Exercises 19 to 40, p. 291, can be studied now.

## Proposition XV. Theorem

**293.** *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*




**Hypothesis.** $\triangle AEB \sim \triangle A'E'B'$; $\triangle EBD \sim \triangle E'B'D'$; $\triangle BCD \sim \triangle B'C'D'$.

The triangles are similarly placed.

**Conclusion.** Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

**Analysis.** The homologous $\angle$s must be proved equal, and the homologous sides must be proved proportional. § 272

**Proof.** 1. $\angle 1 = \angle 1'$; $\angle 2 = \angle 2'$; $\angle 3 = \angle 3'$. Why?

2. $\therefore \angle B = \angle B'$. Why?

3. Similarly, $\angle D = \angle D'$; $\angle E = \angle E'$; $\angle A = \angle A'$; $\angle C = \angle C'$. Prove it.

4. $\dfrac{AE}{A'E'} = \dfrac{BE}{B'E'}$, and $\dfrac{ED}{E'D'} = \dfrac{BE}{B'E'}$. Why?

5. $\therefore \dfrac{AE}{A'E'} = \dfrac{ED}{E'D'}$. Ax. 1, § 51

6. Similarly, $\dfrac{ED}{E'D'} = \dfrac{CD}{C'D'}$.

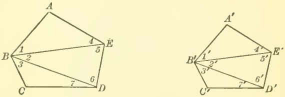
7. Also $\dfrac{AB}{A'B'} = \dfrac{AE}{A'E'}$, and $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$. Why?

8. $\therefore \dfrac{AB}{A'B'} = \dfrac{AE}{A'E'} = \dfrac{ED}{E'D'} = \dfrac{CD}{C'D'} = \dfrac{BC}{B'C'}$. Ax. 1, § 51

9. $\therefore$ Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$. Why?

## Proposition XVI. Problem

**294.** *Upon a given segment, homologous to a given side of a given polygon, construct a polygon similar to the given polygon.*




**Given** polygon $ABCDE$ and segment $A'B'$.

**Required** to construct upon $A'B'$ as side homologous to $AB$ a polygon similar to $ABCDE$.

**Construction.** 1. Divide $ABCDE$ into triangles by drawing $EB$ and $BD$.

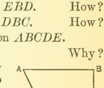
2. Construct $\triangle A'B'E'$ similar to $\triangle ABE$, by making $\angle A' = \angle A$, and $\angle 1' = \angle 1$.

3. Construct $\triangle E'B'D'$ similar to $\triangle EBD$. How?

4. Construct $\triangle D'B'C'$ similar to $\triangle DBC$. How?

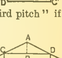
**Statement.** polygon $A'B'C'D'E' \sim$ polygon $ABCDE$. Why?

**Ex. 96.** *ABCD* is the shape of an irregular piece of ground. Make a figure similar to *ABCD* such that each side of the resulting figure shall be three times as long as the corresponding side of the given figure.



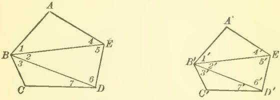

**Ex. 97.** An ordinary shed roof is said to have a "one-third pitch" if the distance $AB$ is one third of the distance $CD$.

A carpenter wishes to order some "two by fours" for the rafters $AE$ of a garage which is to be 24 ft. wide and have a one-third pitch. He allows $\frac{1}{2}$ ft. for cutting at the end $A$, and wants the rafters to project beyond the wall at $C$ so that $CE$ will be 2 ft. What length of "two by fours" must he order if they can be obtained only in even lengths?



## Proposition XVII. Theorem

**295.** *Two similar polygons can be divided into the same number of triangles, similar each to each, and similarly placed.*



**Hypothesis.** Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, homologous vertices being indicated by corresponding letters.

**Conclusion.** The polygons can be divided into the same number of triangles, similar each to each and similarly placed.

**Construction.** Draw $BE$, $BD$, $B'E'$, and $B'D'$.

**Statement.** $\triangle ABE \sim \triangle A'B'E'$; $\triangle EBD \sim \triangle E'B'D'$; $\triangle BCD \sim \triangle B'C'D'$.

**Proof.** 1. $\angle A = \angle A'$ and $\dfrac{AB}{A'B'} = \dfrac{AE}{A'E'}$. Why?

2. $\therefore \triangle BAE \sim \triangle B'A'E'$. Why?

3. $\angle E = \angle E'$ and $\angle 4 = \angle 4'$. Why?

$\therefore \angle 5 = \angle 5'$. Why?

4. $\dfrac{BE}{B'E'} = \dfrac{AE}{A'E'}$, and also $\dfrac{ED}{E'D'} = \dfrac{AE}{A'E'}$. Why?

$\therefore \dfrac{BE}{B'E'} = \dfrac{ED}{E'D'}$. Why?

5. $\therefore \triangle EBD \sim \triangle E'B'D'$. Why?

6. Similarly $\triangle BCD \sim \triangle B'C'D'$.

Note. — If $X$ and $Y$ are any two points of one polygon and $X'$ and $Y'$ are the homologous points of a similar polygon, then $XY$ and $X'Y'$ are homologous segments and $XY : X'Y'$ equals the ratio of similitude.

**296. Fundamental Theorem about Equal Ratios.**

*In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$

then $$\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$$

**Proof.** 1. Let $r=\dfrac{a}{b}$ and hence $br=a$.

2. $\therefore dr=c,\ fr=e,\ hr=g.$ Prove it.

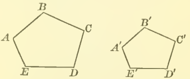
3. $\therefore br+dr+fr+hr=a+c+e+g.$ Why?

4. $\therefore r=\dfrac{a+c+e+g}{b+d+f+h}.$ Prove it.

5. Hence $$\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$$

## Proposition XVIII. Theorem

**297.** *The perimeters of two similar polygons have the same ratio as any two homologous sides.*




**Hypothesis.** $ABCDE$ and $A'B'C'D'E'$ are similar polygons with homologous vertices indicated by corresponding letters.

**Conclusion.**

$$\frac{AB+BC+CD+DE+EA}{A'B'+B'C'+C'D'+D'E'+E'A'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\text{etc.}$$

**Proof.** 1. $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

(Complete the proof. Apply § 296.)

**Note.** — Supplementary Exercises 41 to 44, p. 293, can be studied now.

## SUPPLEMENTARY TOPICS

Five groups of theorems follow, — all of which appear iu modern geometries. It is not necessary, — in fact, it may be unwise to study all of them in every class. The teacher should feel free to select the group or groups which appear to be of most value to the class.

*Each group is independent of each of the others.*

*None of these theorems is required* as an authority in the main lists of theorems of succeeding Books.

Group *A.* — Scales and Scale Drawing.

Group *B.* — Trigonometric Ratios and their Application.

These two groups are interesting and valuable applications of Book III.

Group *C.* — Proportional Division of a Segment.

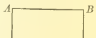
Group *D.* — External and Harmonic Division of a Segment.

Group *E.* — Numerical Relations among Segments of a Triangle.

These last three groups have long appeared in geometries.

## *A.* Scales and Scale Drawing

**298.** Scale drawings are a common and useful application of similar polygons.




The adjoining figure represents a lot $150' \times 275'$. It is drawn to the scale of $1''$ to $200'$; that is, $AB$, $\frac{3}{4}''$ in length, represents $150'$ and $BC$, $1\frac{3}{8}''$ in length, represents $275'$. If the corners of the lot itself are denoted by $A'$, $B'$, $C'$, and $D'$ respectively, then $AB : A'B' = 1 : 2400$, and $BC : B'C = 1 : 2400$.

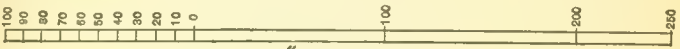
($1''$ to $200'$ is $1''$ to $2400''$ or $1 : 2400$.)

$ABCD$ is similar to $A'B'C'D'$, for the two figures are mutually equiangular (being rectangles) and their homologous sides are proportional (the ratio of similitude being $1 : 2400$).

Since, in similar figures, the ratio of any two homologous sides equals the ratio of similitude (Note, § 295), it is possible to determine from $ABCD$ the approximate length of any segment on the field itself.

**299. Scales.** The construction and use of scale drawings are made easy by the construction in advance of the scale itself, unless such a scale is already at hand.

EXAMPLE. — Below is the scale of $1''$ to $100'$ to measure 350 ft.




Scale $1'' = 100$

The segment extending from the zero mark to any division point represents the number of feet indicated above that point. Notice that the left-hand section is divided into ten equal parts.

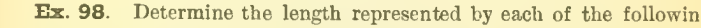
To determine the number of feet represented by a given segment according to the given scale: take the segment on the dividers; place one point of the dividers on a division mark at the right of the zero mark, so that the other point of the dividers will fall on the scale either at the zero mark or to the left of it. The length represented by the segment may then be read to the nearest 5 feet.

Thus, the segment $a$ below represents 235 ft. if drawn to the scale of $1''$ to $100'$.



$a$

**Ex. 98.** Determine the length represented by each of the following segments, assuming that they are drawn to the scale of $1''$ to $100'$.



$a$ $b$ $c$

**Ex. 99.** Determine the approximate number of feet represented by the diagonal $AC$ in the figure of § 298.

**Ex. 100.** Determine the approximate distance of the tree, $T$, from each of the corners of the lot $ABCD$ in the figure of § 298.

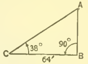
**Ex. 101.** Construct the scale of $1''$ to $1'$ to measure 5 ft., having the left-hand section show the segments corresponding to $2''$, $4''$, etc., to $12''$.

What length, in feet and inches, do segments $a$, $b$, and $c$ of Ex. 98 represent if it is assumed that they are drawn to the scale of $1''$ to $1'$?

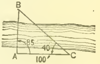
**Ex. 102.** Construct the scale of 1″ to 25′, to measure 100 ft., having the left-hand section show segments corresponding to 5′, 10′, etc., to 25′.

What length do segments $a$, $b$, $c$ of Ex. 98 represent if it is assumed that they are drawn to the scale of 1″ to 25′?

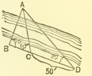
**Ex. 103.** Draw to the scale of 1″ to 25′ the adjoining figure. From the figure so drawn, determine the approximate height of $AB$.




**Ex. 104.** Draw to the scale of 1″ to 25′ a figure similar to the adjoining one. From the resulting figure, determine the approximate distance represented by $AB$.

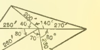



**Ex. 105.** Draw to the scale of 1″ to 25′ a figure similar to the adjoining one. From the resulting figure, determine the approximate length of $AB$, if $CD = 50'$, $\angle ACB = 60°$, $\angle ADC = 30°$, and $\angle ABC = 90°$.



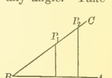

**Ex. 106.** The perimeters of two similar polygons are 119 and 68; if a side of the first is 21, what is the homologous side of the second? (§ 297.)

**Ex. 107.** Draw to the scale of 1″ to 100′ a figure similar to the adjoining one. From the resulting figure, determine the approximate perimeter of the field having the dimensions indicated.




## *B.* Trigonometric Ratios and their Application

**300. Sine of an Angle.** Let $\angle ABC$ be any angle. Take on $BC$ any points $P_1$ and $P_2$. Draw perpendiculars $P_1R_1$ and $P_2R_2$ to $AB$.




Then $\triangle BP_1R_1 \sim \triangle BP_2R_2$. (Why?)

$$\therefore \frac{R_1P_1}{BP_1} = \frac{R_2P_2}{BP_2}.$$

That is, the ratio of the perpendicular $RP$ to the distance $BP$ is the same, regardless of where $P_1$ and $P_2$ are located on $BC$.

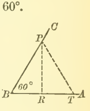
This constant ratio is called the **Sine of** $\angle B$. (**Sin** $B$.)

When the angle is acute, its sides and the perpendicular form a right triangle. In this triangle,

*sine of acute angle = side opposite ÷ hypotenuse.*

The sine of a given angle may be computed as in the following

EXAMPLE. — Let $\angle B = 60°$. Determine sin 60°.



(Why ?)

**Solution.** 1. Draw $PR \perp BA$. Draw $PT = PB$.

2. Then $\triangle PRT \cong \triangle PRB$, and $\angle T = 60°$. (Why ?)

3. $\therefore \triangle PBT$ is equilateral, and $BR = \frac{1}{2} BP$. (Why ?)

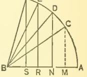
4. Let $BP = 2m$ and hence $BR = m$.

5. In $\triangle BPR$, $RP^2 = 4m^2 - m^2 = 3m^2$. (Why ?)

6. $\therefore RP = m\sqrt{3}$.

7. $\therefore \sin 60° = \dfrac{RP}{BP} = \dfrac{m\sqrt{3}}{2m} = \dfrac{1.732}{2} = .866^{+}$.

**Ex. 108.** Determine as in the example the value of sin 45° and of sin 30°.



**Ex. 109.** Construct a figure similar to the adjoining one making: $\angle ABC = 35°$; $\angle ABD = 50°$; $\angle ABE = 65°$; and $\angle ABF = 75°$. Draw the perpendiculars from $C$, $D$, $E$, and $F$ to $AB$. Measure these perpendiculars and also the radius. Then compute the approximate values of the sine of each of the angles indicated; that is, of sin 35°, sin 50°, sin 65°, and sin 75°.

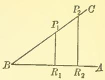
(If you have a metric scale, make $AB = 100$ mm. and measure the perpendiculars in mm.; if you do not have a metric scale, make $AB = 3\frac{1}{4}$ in., and measure the perpendiculars in sixteenths of an inch. If you use metric measures, your sines should be approximately correct to the second decimal place; if you use the English scale, the values should be correct to the first decimal place. Keep your figure for use in a later exercise.)

**Ex. 110.** Construct the acute angle:

(*a*) Whose sine is $\frac{3}{4}$; (*b*) whose sine is $\frac{4}{5}$.

**Ex. 111.** $BC$, 40 mm. long, is on one side of $\angle ABC$, whose sine is $\frac{1}{4}$. How long is $CA$, the perpendicular to side $AB$?

**301. Cosine and Tangent of an Angle.**




It is easily proved that the ratio $\frac{BR}{BP}$ is constant for all positions of $P$ on $BC$, as in § 300; and also that $\frac{RP}{BR}$ is constant.

$\frac{BR}{BP}$ is called the **Cosine of** $\angle B$. (**Cos** $B$.)

$\frac{RP}{BR}$ is called the **Tangent of** $\angle B$. (**Tan** $B$.)

In the right triangle formed when $\angle B$ is an acute angle:

*cosine of acute angle = adjacent side ÷ hypotenuse;*

*tangent of acute angle = opposite side ÷ adjacent side.*

Example. — When $\angle B = 60°$, if $BP = 2m$, then $BR = m$ and $RP = m\sqrt{3}$. (See Example, § 300.)

Hence,
$$\text{Cos } 60° = \frac{m}{2m} = \frac{1}{2} = .500.$$
$$\text{Tan } 60° = \frac{m\sqrt{3}}{m} = \sqrt{3} = 1.732.$$

**Ex. 112.** Compute the cosine and the tangent of 45° and 30° respectively.

**Ex. 113.** In the figure constructed for Ex. 109, measure $BM$, $BN$, $BR$, and $BS$. Then compute the approximate values of:

(*a*) cos 35°; cos 50°; cos 65°; cos 75°;

(*b*) tan 35°; tan 50°; tan 65°; tan 75°.

**302. Table of Values of Trigonometric Ratios.** The values of the sine, cosine, and tangent of certain angles have been computed. See the table opposite.

To determine the sine of 37° from the table: in the first column find 37°; on the same line with it, and in the column headed by the word *Sin*, is found .602. This is the sine of 37°.

Similarly the cosine or tangent of a given angle may be determined from the table.

TABLE OF VALUES OF TRIGONOMETRIC RATIOS

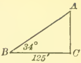
| ANGLE | SIN | COS | TAN | ANGLE | SIN | COS | TAN |
|---|---|---|---|---|---|---|---|
| 10° | .174 | .985 | .176 | 45° | .707 | .707 | 1.000 |
| 11° | .191 | .982 | .194 | 46° | .719 | .695 | 1.036 |
| 12° | .208 | .978 | .213 | 47° | .731 | .682 | 1.072 |
| 13° | .225 | .974 | .231 | 48° | .743 | .669 | 1.111 |
| 14° | .242 | .970 | .249 | 49° | .755 | .656 | 1.150 |
| 15° | .259 | .966 | .268 | 50° | .766 | .643 | 1.192 |
| 16° | .276 | .961 | .287 | 51° | .777 | .629 | 1.235 |
| 17° | .292 | .956 | .306 | 52° | .788 | .616 | 1.280 |
| 18° | .309 | .951 | .325 | 53° | .799 | .602 | 1.327 |
| 19° | .326 | .946 | .344 | 54° | .809 | .588 | 1.376 |
| 20° | .342 | .940 | .364 | 55° | .819 | .574 | 1.428 |
| 21° | .358 | .934 | .384 | 56° | .829 | .559 | 1.483 |
| 22° | .375 | .927 | .404 | 57° | .839 | .545 | 1.540 |
| 23° | .391 | .921 | .424 | 58° | .848 | .530 | 1.600 |
| 24° | .407 | .914 | .445 | 59° | .857 | .515 | 1.664 |
| 25° | .423 | .906 | .466 | 60° | .866 | .500 | 1.732 |
| 26° | .438 | .899 | .488 | 61° | .875 | .485 | 1.804 |
| 27° | .454 | .891 | .510 | 62° | .883 | .469 | 1.881 |
| 28° | .469 | .883 | .532 | 63° | .891 | .454 | 1.963 |
| 29° | .485 | .875 | .554 | 64° | .899 | .438 | 2.050 |
| 30° | .500 | .866 | .577 | 65° | .906 | .423 | 2.144 |
| 31° | .515 | .857 | .601 | 66° | .914 | .407 | 2.246 |
| 32° | .530 | .848 | .625 | 67° | .921 | .391 | 2.356 |
| 33° | .545 | .839 | .649 | 68° | .927 | .375 | 2.475 |
| 34° | .559 | .829 | .675 | 69° | .934 | .358 | 2.605 |
| 35° | .574 | .819 | .700 | 70° | .940 | .342 | 2.747 |
| 36° | .588 | .809 | .727 | 71° | .946 | .326 | 2.904 |
| 37° | .602 | .799 | .754 | 72° | .951 | .309 | 3.078 |
| 38° | .616 | .788 | .781 | 73° | .956 | .292 | 3.271 |
| 39° | .629 | .777 | .810 | 74° | .961 | .276 | 3.487 |
| 40° | .643 | .766 | .839 | 75° | .966 | .259 | 3.732 |
| 41° | .656 | .755 | .869 | 76° | .970 | .242 | 4.011 |
| 42° | .669 | .743 | .900 | 77° | .974 | .225 | 4.331 |
| 43° | .682 | .731 | .933 | 78° | .978 | .208 | 4.705 |
| 44° | .695 | .719 | .966 | 79° | .982 | .191 | 5.145 |
| 45° | .707 | .707 | 1.000 | 80° | .985 | .174 | 5.671 |

**Ex. 114.** Obtain from the table: (*a*) $\sin 40°$; $\sin 53°$; $\sin 26°$; (*b*) $\cos 37°$; $\cos 76°$; $\cos 64°$; (*c*) $\tan 29°$; $\tan 68°$; $\tan 71°$.

**Ex. 115.** What is the angle $x$ if:

(*a*) $\sin x = .454$; $\sin x = .829$; $\sin x = .978$;
(*b*) $\cos x = .956$; $\cos x = .719$; $\cos x = .358$;
(*c*) $\tan x = .424$; $\tan x = 1.111$; $\tan x = 3.732$.

**303. Application of Trigonometric Ratios.**

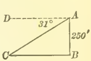



EXAMPLE 1. — Assume that at point $B$ $\angle CBA = 34°$ and that $BC = 125$ ft. How high above ground is point $A$? .

**Solution.** 1. $\frac{AC}{BC} = \tan 34°$, or $AC = BC \times \tan 34°$.

2. $\therefore AC = 125 \times .68 = 85$ ft.

**Note.** — $\angle CBA$ is called the **Angle of Elevation** of $A$ at point $B$.




EXAMPLE 2. — $AB$ represents a lighthouse 250 ft. high. $DA$ is an imaginary line parallel to $BC$. $C$ represents the position of a ship. $\angle DAC = 31°$. How far from $B$ is $C$?

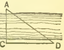
**Solution.** 1. $\angle BCA = 31°$. Why?

2. $\tan 31° = \frac{AB}{BC}$, or $BC = AB \div \tan 31°$.

3. $\therefore BC = 250 \div .601 = 415.9$ ft.

That is, $BC$ is about 416 ft.

**Note.** — $\angle DAC$ is called the **Angle of Depression** of $C$ at $A$.




**Ex. 116.** In the adjoining figure, if $AC \perp CD$, $CD = 100$ ft., and $\angle D = 62°$, what are the distances $AC$ and $AD$?

**Ex. 117.** Find the angle of elevation of the sun when a monument whose height is 360 ft. casts a shadow 400 ft. in length.

**Ex. 118.** Determine the length of one side of an equilateral polygon having nine sides which is inscribed in a circle of radius 10 ft.

Determine the distance of each of the sides from the center of the circle.

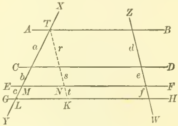
**Ex. 119.** A boy flying a kite knows that he has 500 ft. of twine. Another boy stations himself approximately below the kite. The angle made by the string with an imaginary line running from the first boy to the second is approximately 50°. Determine the approximate height of the kite.

**Ex. 120.** At a point 169 ft. from the foot of a tower surmounted by a pole, the angle of elevation of the top of the tower is 35° and that of the top of the pole is 47°. Find the length of the pole.

## *C.* Proportional Division of a Segment

### Proposition XIX. Theorem

**304.** *Parallel lines intercept proportional segments on all transversals.*



**Hypothesis.** Parallels $AB$, $CD$, $EF$, and $GH$ intercept segments $a$, $b$, and $c$, on $XY$ and $d$, $e$, and $f$ on $ZW$, respectively.

**Conclusion.** $\frac{a}{d}=\frac{b}{e}=\frac{c}{f}.$

*Suggestions.* — 1. Draw $TK \parallel ZW$. Let the ‖s intercept the segments $r$, $s$, and $t$ on $TK$.

2. Compare $r$, $s$, and $t$, with $d$, $e$, and $f$, respectively.
3. In $\triangle\ TKL$, compare $a : r$ with $TL : TK$.
4. In $\triangle\ TKL$, compare $c : t$ with $TL : TK$.
5. In $\triangle\ TMN$, compare $a : r$ with $b : s$.
6. Then compare $a : r$, $c : t$, and $b : s$.

Complete the proof, using the facts obtained in step 2.

**Ex. 121.** Divide a segment into parts proportional to any number of given segments.

**Given** segment $AB$, and segments $m$, $n$, and $p$.

**Required** to divide $AB$ into segments $x$, $y$, and $z$, so that

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{p}.$$

*Suggestion.*— Base the construction on Proposition XIX.

**Ex. 122.** Divide a segment 6 in. in length into segments proportional to 2, 3, and 4.

**Ex. 123.** Construct a triangle similar to a given triangle, having given its perimeter.

**Ex. 124.** A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals.

**Note.** — Supplementary Exercises 45 to 46, p. 293, can be studied now.

## *D.* External and Harmonic Division of a Segment

**305. External Division of a Segment.** If $P$ is a point on line $AB$ but not located between $A$ and $B$, then $P$ **divides** $AB$ **externally** into segments $AP$ and $PB$.

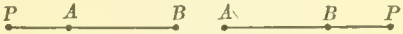



FIG. 1

The following justification is given for calling $AP$ and $PB$ segments of $AB$. Direction on a segment may be indicated by reading the segment from its beginning point to its end point; thus

$$AB = A \longrightarrow B, \text{ and } BA = A \longleftarrow B.$$

Direction from left to right is regarded as positive and from right to left as negative. Hence $BA = -AB$ and $AB + BA = 0$.

Clearly then the algebraic sum of $AP$ and $PB$ in Fig. 1 equals $AB$, for $AP + PB = AP + PA + AB$ and $AP + PA = 0$.

The consequence is that the algebraic sum of $AP$ and $PB$ is $AB$, no matter where $P$ is located on the line $AB$. (See figures below.)


FIG. 2

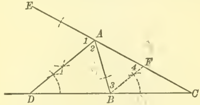
**Note.** — Prove that $AP + PB = AB$ in each case in Fig. 2.




**Ex. 125.** What segment represents the algebraic sum of: (*a*) $AB + BC$? (*b*) $AD + DC$? (c) $BA + AC + CB$?

PROPOSITION XX. THEOREM

**306.** *In any triangle, the bisector of an exterior angle at any vertex divides the opposite side externally into two segments whose ratio equals the ratio of the two adjacent sides of the triangle.*




**Hypothesis.** $AD$ bisects ext. $\angle BAE$ of $\triangle ABC$, meeting $CB$ extended at $D$.

**Conclusion.** $BD : DC = BA : AC$.

**Proof.** 1. Draw $BF \parallel DA$, meeting $AC$ at $F$.

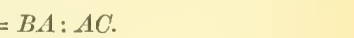
2. Then $BD : DC = FA : AC$.

It remains to prove that $FA = BA$. Try to prove that $\angle 3 = \angle 4$, using the hypothesis and construction. Proof left to the pupil.

**Note.** — The converse of Proposition XX is also true. It may be proved by laying off $AF = AB$.

**Ex. 126.** The sides of a triangle are $AB = 5$, $BC = 7$, and $CA = 8$; find the segments into which side 8 is divided by the bisector of the exterior angle at the opposite vertex.

**307.** A **segment** is **divided harmonically** if it is divided internally and externally into segments having the same ratio.

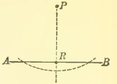



Thus, if $AB = 1$, $AX = \frac{3}{4}$, and $AY = \frac{3}{2}$, then $\frac{AX}{XB} = \frac{AY}{YB}$, since each ratio equals $\frac{3}{1}$. Hence $X$ and $Y$ divide $AB$ harmonically.

**Ex. 127.** Prove that the bisector of an interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side harmonically.

**Ex. 128.** If $X$ and $Y$ divide $AB$ harmonically, then $A$ and $B$ divide $XY$ harmonically. (Use § 253. Verify afterwards for the figure in § 307.)

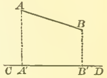
## *E*. Numerical Relations among Segments of a Triangle





**308.** The **Projection of a Point** upon a given line is the foot of the perpendicular drawn from the point to the line.

Thus, $R$ is the projection of $P$ on $AB$.





**309.** The **Projection of a Segment** upon a given line is the distance between the projections of its end-points.

Thus, the projection of $AB$ on line $CD$ is $A'B'$.

The symbol "$p_{CD}^{AB}$" is read "the projection of $AB$ on $CD$."

**Ex. 129.** Draw a segment $AB$ and also four straight lines not parallel to $AB$ but also not *crossing* $AB$.

(*a*) Determine the projection of $AB$ on each of the straight lines.

(*b*) Are the projections all of the same length?

**Ex. 130.** Draw an acute scalene triangle. Show by means of a drawing the projection of the shortest side upon each of the other sides.

**Ex. 131.** Repeat the preceding exercise for an obtuse triangle.

**Ex. 132.** Draw an obtuse triangle. Obtain the projection of the longest side upon each of the other sides.

**Ex. 133.** If $AB$, extended, makes an acute angle with a line $m$, prove that $p_m^{AB}$ is less than $AB$.

**Ex. 134.** If $AB \parallel m$, how does $p_m^{AB}$ compare with segment $AB$?

**Ex. 135.** If $AB \perp m$, what is the length of $p_m^{AB}$?

**Ex. 136.** What part of the base of an isosceles triangle is the projection upon the base of one of the equal sides?

**Ex. 137.** If the equal sides of an isosceles trapezoid be projected upon the lower base, the projections are equal.

**Ex. 138.** Draw a right triangle and draw the median to the hypotenuse. Prove that the projection of the median upon either leg of the triangle is one half of that leg.

### Proposition XXI. Theorem

**310.** *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other upon it.*

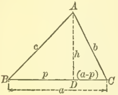



FIG. 1

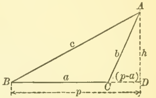



FIG. 2

**Hypothesis.** In $\triangle ABC$, $\angle B$ is acute.

**Conclusion.** $b^2 = a^2 + c^2 - 2\, a \cdot p_a^c$.

**Proof.** 1. Draw $AD \perp BC$. Then $BD = p_a^c = p$.

2. In Fig. 1, $b^2 = h^2 + \overline{DC}^2$. Why?

3. But $DC = a - p$, and $h^2 = c^2 - p^2$. Why?

4. $\therefore\ b^2 = c^2 - p^2 + (a - p)^2$. Ax. 2, § 51

(Complete the proof.)

**Note 1.** — A similar proof may be given from Fig. 2. In Fig. 2, $DC = p - a$.

**Note 2.** — The conclusion of Proposition XXI is a formula connecting the three sides of a triangle with the projection of one side upon one of the other two sides. Altogether four different numbers are involved. Hence, when three of these numbers are known, the fourth may be determined by substituting in the formula and solving the resulting equation. In the right member, there appear the squares of two sides and the projection of one of these upon the other; in the left member, there appears the square of the third side.

**Ex. 139.** Determine:

(*a*) $p_b^a$ when $a = 13$, $b = 14$, and $c = 15$.

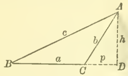
(*b*) $b$ when $a = 10$, $c = 12$, and $p_a^c = 9$.

(*c*) $c$ when $a = 11$, $b = 16$, and $p_a^c = 7$.

(*d*) $a$ when $b = 18$, $c = 12$, and $p_a^c = 4$.

### PROPOSITION XXII. THEOREM

**311.** *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



**Hypothesis.** In $\triangle ABC$, $\angle C$ is an obtuse $\angle$.

**Conclusion.** $c^2 = a^2 + b^2 + 2\,a \cdot p_a^b$.

**Proof.** 1. Draw $AD \perp BC$ extended. Then $p_{BC}^b = CD = p$.

2. Then in $\triangle ABD$, $c^2 = h^2 + \overline{BD}^2$. Why?

3. But $h^2 = b^2 - p^2$; and $BD = a + p$.

4. $\therefore\ c^2 = (b^2 - p^2) + (a + p)^2$.

(Complete the proof.)

**312. Cor.** If $a$, $b$, and $c$ are the sides of a triangle:

$\angle A$ is acute if $a^2 < b^2 + c^2$;
$\angle A$ is obtuse if $a^2 > b^2 + c^2$;
$\angle A$ is a right $\angle$ if $a^2 = b^2 + c^2$.
(Proof is indirect in each case.)

**Ex. 140.** Is the greatest angle of the triangle whose sides are 8, 9, and 12 acute, right, or obtuse?

**Ex. 141.** Is the greatest angle of the triangle whose sides are 12, 35, and 37 acute, right, or obtuse?

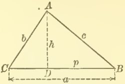
**Ex. 142.** Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides of the parallelogram. (Use § 310 and § 311.)

**Ex. 143.** If $AB$ and $AC$ are the equal sides of an isosceles triangle, and $BD$ is drawn perpendicular to $AC$, prove

$$2\,AC \times CD = \overline{BC}^2.$$

**Note.** — Supplementary Exercises 47 to 51, p. 293, can be studied now.

**313.** When the three sides of a triangle are known, the altitude to each side can be computed.



1. Assume $AD = h_a$, and $\angle B$ to be an acute angle.
2. $\therefore b^2 = a^2 + c^2 - 2\,a \cdot p_a^c$ or $b^2 = a^2 + c^2 - 2\,ap$.
3. $$\therefore p = \frac{a^2 + c^2 - b^2}{2\,a}.$$
4. $h_a^2 = c^2 - p^2 = (c + p)(c - p)$.
5. $\therefore h_a^2 = \left[c + \frac{a^2 + c^2 - b^2}{2\,a}\right]\left[c - \frac{a^2 + c^2 - b^2}{2\,a}\right]$
6. $= \left[\frac{2\,ac + a^2 + c^2 - b^2}{2\,a}\right]\left[\frac{2\,ac - a^2 - c^2 + b^2}{2\,a}\right]$
7. $= \frac{[(a + c)^2 - b^2][b^2 - (a - c)^2]}{4\,a^2}$
8. $= \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4\,a^2}.$
9. Let $a + b + c = 2\,s.$
10. $\therefore a + b - c = 2\,s - 2\,c = 2(s - c).$

Similarly, $b + c - a = 2(s - a)$; and $c + a - b = 2(s - b)$.

11. $\therefore h_a^2 = \frac{2\,s \cdot 2(s - b) \cdot 2 \cdot (s - c) \cdot 2(s - a)}{4\,a^2}$

$$= \frac{4\,s(s - a)(s - b)(s - c)}{a^2}.$$

12. $\therefore h_a = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}$

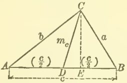
Similarly, $$h_b = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)};$$

and $$h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

**Ex. 144.** Find the three altitudes of the triangle whose sides are 13, 14, and 15, getting the results correct to one decimal place.

### PROPOSITION XXIII. THEOREM

**314.** *In any triangle, the sum of the squares of two sides equals twice the square of half the third side plus twice the square of the median drawn to that side.*



**Hypothesis.** In $\triangle ABC$, $CD$ is the median to side $AB$.

**Conclusion.** $$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m_c^2.$$

**Note.** — $\angle ADC$ is either a rt. $\angle$, an acute $\angle$, or an obtuse $\angle$. When it is a rt. $\angle$, the proof is quite easy.

**Proof.** 1. Assume that $\angle ADC$ is obtuse and hence $\angle BDC$ is acute.

2. Draw $CE \perp AB$, so that $DE = p_{AB}^{CD}$.

(Complete the proof.)

*Suggestion.* — Determine $a^2$ from $\triangle BCD$ by § 310; $b^2$ from $\triangle ACD$ by § 311; then add, so as to obtain $a^2 + b^2$.

**Note.** — By Proposition XXIII, it is possible to determine the three medians of a triangle when the three sides of the triangle are known.

**315. Cor.** *The difference between the squares of two sides of a triangle equals twice the product of the third side and the projection of the median upon that side.*

**Con.** $$b^2 - a^2 = 2\,c \cdot p_c^{m_c}.$$

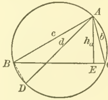
*Suggestion.* — Determine $b^2$ and $a^2$ and then subtract the value of $a^2$ from that of $b^2$.

**Ex. 145.** Determine $m_a$ when $b = 12$, $c = 16$, and $a = 20$. Determine also $m_b$ and $m_c$.

**Ex. 146.** From the conclusion of § 314 derive a formula for $m_c$ in terms of $a$, $b$, and $c$.

PROPOSITION XXIV. THEOREM

**316.** *In any triangle, the product of two sides equals the product of the diameter of the circumscribed circle and the altitude upon the third side.*



**Hypothesis.** $\triangle ABC$ is inscribed in $\odot\ O$; $AD$ is a diameter of $\odot\ O$; $AE = h_a$.

**Conclusion.** $b \cdot c = d \cdot h_a$.

Analysis and proof left to the pupil. See analysis of § 283.

**317.** By § 313, $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

Hence, by § 316, $b \cdot c = \frac{2\,d}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore\ 2\,d\sqrt{s(s-a)(s-b)(s-c)} = abc,$$

or
$$d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

Hence, when the sides of a triangle are known, the diameter of the circumscribed circle can be computed.

**Ex. 147.** Determine the diameter of the circle circumscribed about the triangle whose sides are 13, 14, and 15.

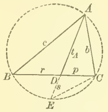
**Ex. 148.** If two adjacent sides and one of the diagonals of a parallelogram are 7, 9, and 8 respectively, find the other diagonal. (§ 314.)

**Ex. 149.** The sides $AB$ and $AC$ of $\triangle ABC$ are 16 and 9 respectively, and the length of the median drawn from $C$ is 11. Find side $BC$. (§ 314.)

**Ex. 150.** If the sides of $\triangle ABC$ are 10, 14, and 16, find the lengths of the three medians. Determine also the diameter of the circumscribed circle.

### Proposition XXV. Theorem

**318.** *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*




**Hypothesis.** $AD$ bisects $\angle A$ of $\triangle ABC$, meeting $BC$ at $D$. (Let $BD = r$, and $DC = p$.)

**Conclusion.** $b \cdot c = t_A^2 + r \cdot p$.

**Proof.** 1. Circumscribe a circle about $\triangle ABC$.
Extend $AD$ to meet the circle at $E$. Draw $CE$.

2. $\triangle ABD \sim \triangle ACE$. Give full proof.

3. $\therefore \dfrac{c}{AE} = \dfrac{t_A}{b}$, or $bc = AE \cdot t_A$. Give full proof.

4. $\therefore bc = t_A \cdot (t_A + s)$. $(AE = t_A + s)$

5. $\therefore bc = t_A^2 + t_A \cdot s$.

6. But $t_A \cdot s = r \cdot p$. § 283

7. $\therefore bc = t_A^2 + r \cdot p$.

**Note.** — This proposition makes it possible to compute the bisectors of the three angles of a triangle when the three sides of the triangle are known.

**Ex. 151.** If $c = 4$, $b = 5$, and $a = 6$, find $t_A$.

*Suggestions.* — 1. It is necessary to find $r$ and $p$ first. This may be done by using § 270.

$$\frac{r}{6-r} = \frac{4}{5}. \quad \text{Whence } r = ? \text{ and } p = 6 - r = ?$$

2. Then substitute in the conclusion of § 318.

**Note.** — Supplementary Exercises 52 to 56, p. 294, can be studied now.

### Miscellaneous Exercises

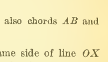
**Ex. 152.** The vertices of quadrilateral $ABCD$ are joined to a point $O$ lying outside the quadrilateral. Points $A'$, $B'$, $C'$, and $D'$ are taken on $OA$, $OB$, $OC$, and $OD$, respectively, so that $A'B' \parallel AB$, $B'C' \parallel BC$, and $C'D' \parallel CD$. Prove $A'D' \parallel AD$.

**Ex. 153.** Two circles are tangent externally at point $C$. Through $C$, a straight line is drawn, meeting the first circle at $A$ and the second at $D$; another straight line through $C$ meets the first circle at $B$ and the second at $E$. Prove $AC : CD = BC : CE$.

*Suggestion.* — Draw the common tangent at $C$, and also chords $AB$ and $ED$.

**Ex. 154.** If $P$ and $S$ are two points on the same side of line $OX$ such that the perpendiculars $PR$ and $ST$ drawn to $OX$ have the same ratio as $OR$ and $OT$, then points $O$, $P$, and $S$ lie in a straight line.

*Suggestion.* — Prove $\angle ROP = \angle TOS$ by proving $\triangle OPR \sim \triangle OST$.




**Ex. 155.** If two parallels are cut by three or more straight lines passing through a common point, the corresponding segments are proportional.

Prove $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$.

**Ex. 156.** If three transversals intercept proportional lengths on two parallels, the transversals meet at a point.

*Suggestion.* — Let $A'A$ and $B'B$ meet at $O$ and draw $OC$ and $OC'$; then prove $\triangle OBC$ and $OB'C'$ similar. (Fig. Ex. 155.)

**Ex. 157.** Derive a formula for the altitude to the base of an isosceles triangle if the base is $b$ and the equal sides are each $a$. By means of the formula determine the altitude when: (*a*) $a = 12$ and $b = 6$; (*b*) $a = 15$ and $b = 7$.

**Ex. 158.** Find the length of the common external tangent to two circles whose radii are 11 and 18, if the distance between their centers is 25.

*Suggestion.* — See the figure for Problem 2, § 236.

**Ex. 159.** If $BE$ and $CF$ are the medians drawn from the extremities of the hypotenuse of right triangle $ABC$, prove $4\,\overline{BE}^2 + 4\,\overline{CF}^2 = 5\,\overline{BC}^2$.

**Ex. 160.** Prove that the projections of two parallel sides of a parallelogram upon either of the other sides are equal.

**Ex. 161.** $BC$ is the base of an isosceles triangle $ABC$ inscribed in a circle. If a chord $AD$ is drawn, cutting $BC$ at $E$, prove $\overline{AB}^2 = \overline{AE}^2 + BE \times CE$.

*Suggestions.* — 1. The proof is like that for § 318.
2. Prove $\triangle ABD \sim \triangle ABE$.

**Ex. 162.** Prove that the non-parallel sides of a trapezoid and the line joining the middle points of the parallel sides, if extended, meet in a common point.

### Review Questions

1. What is meant by taking a proportion by (*a*) inversion? (*b*) composition? (*c*) alternation?

2. Complete the following theorem: "If the product of two numbers equals the product of two other numbers, one pair, ..."

3. Define: (*a*) mean proportional; (*b*) fourth proportional; (*c*) similar polygons; (*d*) ratio of similitude.

4. Are mutually equiangular triangles similar?
Are mutually equiangular polygons similar?

5. State all of the theorems by which two triangles can be proved similar.

6. How do you select the homologous sides of similar triangles? What do you know about them?

7. What do you know about the ratio of homologous altitudes of similar triangles?
What about the ratio of the perimeters of similar triangles?
What about the ratio of the areas of similar triangles?

8. What is the Pythagorean theorem?

9. Find the mean proportional between 5 and 15.
Construct the mean proportional between segments $r$ and $s$.

10. What are the two conclusions which follow from the hypothesis that the altitude is drawn to the hypotenuse of a right triangle?

# BOOK IV

## AREAS OF POLYGONS

**319.** A polygon, being a closed line (§ 7), incloses a *limited portion of the plane.*

In measurement theorems, the words "rectangle," "parallelogram," "polygon," etc., mean the surface within the figure mentioned.

**320.** The **Area** of the surface within a closed line is the ratio of the surface to the unit of surface measure.

Thus, in the adjoining figure, if the unit of surface is one small square, the area of the rectangle is 30.

*A*

It has become customary, when speaking of the area of a figure, to mention at once the unit of surface; thus, in the foregoing example, it is customary to say that the area is 30 small squares. Remember, however, that *the area is 30.*

**321.** The usual **Unit of Surface** is a square whose side is some linear unit: as, a square inch or a square centimeter. In this text, it will be assumed that the unit always is such a square unit.

**Ex. 1.** In the following figures, assume that the unit of surface is a small square. (*a*) What is the exact area of Figs. 1 and 2?

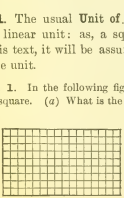


FIG. 1

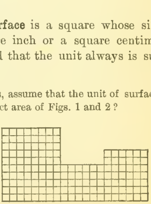


FIG. 2

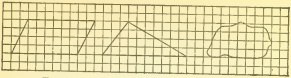


FIG. 3 FIG. 4 FIG. 5

(b) What is the approximate area of Figs. 3, 4, and 5?

(Include the square in the area if half or more than half of it lies within the figure; do not include it otherwise.)

**322.** Two limited portions of a plane are **Equal** (=) if their areas are equal when they are measured by the same unit.

Since the test of the equality of two figures is the equality of two numbers, the usual axioms apply when equal figures are added or subtracted, or when they are multiplied or divided by the same number.

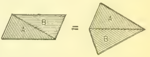
Thus, if equal figures are added to equal figures, the sums are equal; also halves of equal figures are equal.

**Ex. 2.** Of the figures in Ex. 1, are any two or more equal?

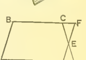
**323.** *Two congruent figures are necessarily equal, but two equal figures are not necessarily congruent.*

Also, two figures which consist of parts which are respectively congruent are equal.

Thus, the parallelogram and the kite-shaped figure made from it by placing the two triangles together as in the figure adjoining are equal.



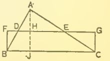

**Ex. 3.** If $E$ is the mid-point of one of the non-parallel sides of trapezoid $ABCD$, and a parallel to $AB$ drawn through $E$ meets $BC$ extended at $F$ and $AD$ at $G$, prove that parallelogram $ABFG$ is equal to trapezoid $ABCD$.

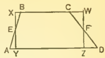



*Suggestion.* — Prove $\triangle CEF \cong \triangle GED$, and apply § 323.

**Ex. 4.** In the adjoining figure, $D$ and $E$ are the mid-points of $AB$ and $AC$; $AJ \perp BC$; $BF \perp DE$ extended at $F$; $CG \perp DE$ extended at $G$. Prove that $\triangle ABC$ equals $\square\ BFGC$.

*Suggestion.* — Prove $\triangle BDF \cong \triangle DAH$, and $\triangle CEG \cong \triangle AEH$.






**Ex. 5.** In the adjoining figure, $E$ and $F$ are the mid-points of sides $AB$ and $CD$ of trapezoid $ABCD$; $XY$ and $ZW$ are drawn through $E$ and $F$ respectively $\perp AD$, meeting $BC$ extended at $X$ and $W$ respectively. Prove that $XYWZ = ABCD$.

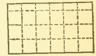
**Ex. 6.** Let $K$ be the mid-point of side $BC$ and $H$ the mid-point of side $AD$ of $\square ABCD$; let $FE$, drawn through the mid-point $G$ of $KH$, intersect $BC$ and $AD$ at $F$ and $E$ respectively. Prove that $FE$ divides $ABCD$ into two equal quadrilaterals.

**Note.** — Supplementary Exercises 1-3, p. 294, can be studied now.

## MEASUREMENT OF RECTANGLES

**324.** The **Dimensions** of a rectangle are the **Base** and **Altitude**.

**325. Area of a Rectangle.** (Informal treatment.) If the base of a rectangle measures 6 and its altitude 5 linear units, the area is evidently $6 \times 5$ or 30 surface units.

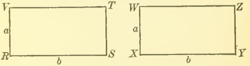


If the base measures 6 units and the altitude measures $3\frac{1}{2}$ units, the area is evidently $6 \times 3.5$ or 21 surface units.

These two examples suggest the theorem:

*The number of surface units in the area of a rectangle is the product of the number of linear units in its base and the number in its altitude.* More briefly, this theorem is expressed: *the area of a rectangle is the product of its base and its altitude.*

The theorem is proved in the following three propositions.

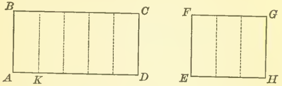



**326. Comparison of Rectangles.** Rectangles may be compared without computing their areas.

*Two rectangles having equal bases and altitudes are equal,* for it is evident that they can be made to coincide by superposition.

## Proposition I. Theorem

**327.** *Two rectangles having equal altitudes are to each other as their bases.*



**Hypothesis.** Rectangles $ABCD$ and $EFGH$ have equal altitudes $AB$ and $EF$, and bases $AD$ and $EH$, respectively.

**Conclusion.** $$\frac{ABCD}{EFGH} = \frac{AD}{EH}.$$

Case I. Assume that $AD$ and $EH$ are commensurable.

§ 211

**Proof.** 1. Let $AK$, a common measure of $AD$ and $EH$, be contained in $AD$ 5 times and in $EH$ 3 times. Draw ⊥s to $AD$ and $EH$ at the points of division.

2. Then $ABCD$ and $EFGH$ are divided into equal rectangles. Why?

(Complete the proof.)

*Suggestions.* — What is the value of $\frac{AD}{EH}$? of $\frac{ABCD}{EFGH}$? Then compare these ratios.

Case II. When $AD$ and $EH$ are incommensurable, the theorem is still true. The proof is given in § 425.

**328. Cor.** *Two rectangles having equal bases are to each other as their altitudes.*

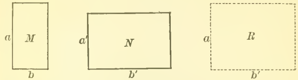
**Ex. 7.** Construct a rectangle which will be three times a given rectangle; also one which will be three fourths a given rectangle.

**Ex. 8.** Two rectangles $M$ and $T$ have equal bases $b$ and altitudes $r$ and $s$ respectively. What is the ratio of $M$ to $T$?

**Note.** — Supplementary Exercises 4 to 5, p. 295, can be studied now.

## Proposition II. Theorem

**329.** *Two rectangles are to each other as the products of their bases by their altitudes.*




**Hypothesis.** Rectangle $M$ has base $b$ and altitude $a$; rectangle $N$ has base $b'$ and altitude $a'$.

**Conclusion.** $\frac{M}{N}=\frac{ab}{a'b'}$.

**Proof.** 1. Let rectangle $R$ have base $b'$ and altitude $a$.

2. $\therefore \frac{M}{R}=\frac{b}{b'}$, since $M$ and $R$ have equal altitudes. Why?

3. Also $\frac{R}{N}=\frac{a}{a'}$, since $R$ and $N$ have equal bases. Why?

4. $\therefore \frac{M}{R}\times\frac{R}{N}=\frac{ab}{a'b'}$, or $\frac{M}{N}=\frac{ab}{a'b'}$. Why?

**Ex. 9.** What is the ratio of rectangles $R$ and $S$ if their dimensions are as follows?

(A)

| | $R$ | $S$ |
|---|---|---|
| Altitude | $k$ | $x$ |
| Base | $w$ | $y$ |

(B)

| | $R$ | $S$ |
|---|---|---|
| Altitude | 4 | 5 |
| Base | 10 | 16 |

(C)

| | $R$ | $S$ |
|---|---|---|
| Altitude | 12 | 15 |
| Base | 20 | 18 |

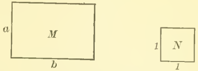
**Ex. 10.** If $R$, $S$, $T$, and $X$ are rectangles having the dimensions indicated in the adjoining table, determine the ratio of each rectangle to each of the others.

(Thus, determine $R : S$, $R : T$, etc.)

| | Altitude | Base |
|---|---|---|
| $R$ | 10 | 6 |
| $S$ | 5 | 12 |
| $T$ | 10 | 8 |
| $X$ | 10 | 12 |

## Proposition III. Theorem

**330.** *The area of a rectangle is the product of its base and altitude.*



**Hypothesis.** Rectangle $M$ has altitude $a$ and base $b$.

**Conclusion.** Area of $M = ab$.

**Proof.** 1. Let square $N$ be the unit of surface measure.

2. Then area of $M =$ the ratio of $M$ to $N$. § 320

3. $\dfrac{M}{N} = \dfrac{ab}{1 \times 1}$. Why?

4. $\therefore$ area of $M = ab$.

**Note.** — Remember that this theorem means that the number of square units in the area equals the product of the number of linear units in the base by the number in the altitude. A similar interpretation must be given for each of the measurement theorems of this Book.

**Ex. 11.** A business corner 50 ft. × 120 ft. is valued at $9000. What is the value per square foot?

**Ex. 12.** The area of a square is 590.49 sq. ft. Find its perimeter.

**Ex. 13.** A rectangle has the dimensions 30 ft. and 120 ft. Compare its perimeter with that of an equal square.

**Ex. 14.** An ordinary eight-room house costs approximately $4.75 per square foot of ground covered by it. What is the approximate cost of a house 27 ft. × 36 ft.?

**Ex. 15.** The area of a rectangle is 147 sq. ft. Its base is three times its altitude. What are its dimensions?

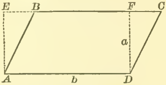
**Ex. 16.** What are the dimensions of a rectangle whose area is 168 sq. ft. and whose perimeter is 58 ft.?

*Suggestion.* — Let the base $= x$ and the altitude $= y$. Form two equations and complete the solution algebraically.

**Ex. 17.** What is the length of the diagonal of a rectangle whose area is 2640 sq. ft., if its altitude is 48 ft.?

### Proposition IV. Theorem

**331.** *The area of a parallelogram equals the product of its base and altitude.*



**Hypothesis.** $ABCD$ is a parallelogram.
Its altitude $DF = a$; its base $AD = b$.

**Conclusion.** Area of $ABCD = ab$.

**Proof.** 1. Draw $AE \parallel DF$, meeting $BC$ extended at $E$.
2. $\therefore AEFD$ is a rectangle. Why?
3. $\triangle AEB = \triangle FCD$. Give the full proof.
4. $\therefore \square ABCD = \square AEFD$. Why?
5. But area of $\square AEFD = ab$. Why?
6. $\therefore$ area of $ABCD = ab$. Ax. 1, § 51

**332. Corollaries.** Let $\square P_1$ have base $b_1$ and altitude $a_1$; and $\square P_2$ have base $b_2$ and altitude $a_2$.

(1) *Parallelograms having equal bases and equal altitudes are equal.*

(2) *Two parallelograms are to each other as the products of their bases by their altitudes.*

For, since $\square P_1 = a_1b_1$ and $\square P_2 = a_2b_2$, then $\dfrac{\square P_1}{\square P_2} = \dfrac{a_1b_1}{a_2b_2}$.

(3) *Parallelograms having equal altitudes are to each other as their bases.*

For, in (2), if $a_1 = a_2$, then $\square P_1 : \square P_2 = b_1 : b_2$.

(4) *Parallelograms having equal bases are to each other as their altitudes.*

**Ex. 18.** What is the area of ▱ $R$, of ▱ $S$, and of ▱ $T$?

(*a*) ▱ $R$ has altitude 4 in. and base 9 in.

(*b*) ▱ $S$ has altitude 15 ft. and base 20 ft.

(*c*) ▱ $T$ has altitude $3x$ in. and base $11y$ in.

**Ex. 19.** What is the altitude of a parallelogram whose area is 56 sq. in., if its base is 14 in.?

**Ex. 20.** Construct a parallelogram equal to twice a given parallelogram.

**Ex. 21.** Construct a rectangle equal to two thirds a given parallelogram.

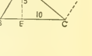
**Ex. 22.** Divide a parallelogram into two equal parallelograms; into four equal parallelograms.

**Ex. 23.** What is the ratio of ▱ $P$ to ▱ $R$ if the base of each is 10 in. and the altitudes are 5 in. and 8 in. respectively?

**Ex. 24.** Construct a ▱ $ABCD$ having $AB = 3$ in. and $BC = 4$ in., and having: (*a*) $\angle B = 30°$; (*b*) $\angle B = 45°$. (*c*) Determine the area of each of the parallelograms.

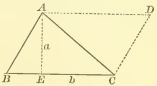
**Ex. 25.** The base of $\triangle ABC$ is 10 and the altitude is 5. What is the area of $\triangle ABC$?

*Suggestion.* — Draw $AD \parallel BC$ and $CD \parallel AB$ to form ▱ $ABCD$. Compare $\triangle ABC$ with ▱ $ABCD$. Then determine the area of ▱ $ABCD$ and finally of $\triangle ABC$.




## Proposition V. Theorem

**333.** *The area of a triangle equals one half the product of its base and altitude.*




**Hypothesis.** $\triangle ABC$ has altitude $AE = a$ and base $BC = b$.

**Conclusion.** Area of $\triangle ABC = \frac{1}{2}ab$.

[Proof to be given by the pupil.]

*Suggestion.* — Construct ▱ $ABCD$ and proceed as in Ex. 25.

**334. Corollaries.** By proofs similar to those given in § 332, it follows that:

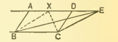
(1) *Triangles having equal bases and equal altitudes are equal.*

(2) *Two triangles are to each other as the products of their bases by their altitudes.*

(3) *Triangles having equal altitudes are to each other as their bases.*

(4) *Triangles having equal bases are to each other as their altitudes.*

(5) *A triangle is one half a parallelogram having the same base and altitude.*




**Ex. 26.** (*a*) Compare $\square ABCD$ with $\triangle BCE$.

(*b*) Compare $\triangle BCX$ with $\triangle BCE$.

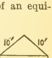
(*c*) If $X$ is the mid-point of $AD$, compare $\triangle ABX$ with $\triangle XCD$; also compare $\triangle XCD$ with $\triangle BCE$.

**Ex. 27.** Determine the area of an isosceles right triangle whose leg is 9 in.

**Ex. 28.** Determine the area of an equilateral triangle whose side is 10 in.

**Ex. 29.** (*a*) Prove that the area of an equilateral triangle whose side is $s$ is $\frac{s^2}{4}\sqrt{3}$. (Memorize this formula.)

(*b*) Using the formula developed in (*a*), obtain the area of an equilateral triangle whose side is: (1) 12 in.; (2) 15 in.




**Ex. 30.** Find the area of the front of the garage whose dimensions are indicated in the adjoining figure.

**Ex. 31.** What is the area of the rhombus whose diagonals are 10 and 16 respectively?

**Ex. 32.** What is the length of the side of a square whose area equals that of a triangle whose base is 24 and whose altitude is 12?

**Ex. 33.** If $BD$ is the median to side $AC$ of $\triangle ABC$, prove that $\triangle ABD = \triangle BDC$. (Draw the altitude $BF$ to side $AC$.)

**Ex. 34.** Prove that the diagonals of a parallelogram divide the parallelogram into four equal triangles.

**Ex. 35.** If segments are drawn from two opposite vertices of a quadrilateral to the mid-point of the diagonal joining the other two vertices, the broken line so formed divides the quadrilateral into two equal parts.

**Ex. 36.** Through the vertex $A$ of $\triangle ABC$, draw a line $MN$ parallel to $BC$. On $MN$ take any point $X$ and prove that $\triangle XBC = \triangle ABC$.

**Ex. 37.** Construct a triangle twice as large as a given triangle:

(*a*) having the same base as the given triangle;

(*b*) having the same altitude as the given triangle.

**Ex. 38.** Construct a rectangle equal to a given triangle.

**Ex. 39.** Construct a triangle equal to a given rectangle.

**Ex. 40.** Construct a right triangle equal to a given triangle and having the same base as the triangle.

**Ex. 41.** Construct an isosceles triangle equal to a given triangle and having the same base as the given triangle.

### 335. The Area of a Triangle can be expressed in Terms of its Sides.

**Solution.** 1. If $a$, $b$, and $c$ are the sides of $\triangle ABC$, and $s = \frac{1}{2}(a+b+c)$, it can be proved that the altitude drawn to side $a$ is given by the formula:

$$h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}. \qquad \S 313$$

**Note.** — The proof may be read if desired. Often in mathematics, we use provable formulæ which we may not have proved ourselves.

2. Area of $\triangle ABC = \frac{1}{2} a \cdot h_a$.

3. $\therefore$ area of $\triangle ABC = \frac{1}{2} a \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$,

4. or area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Remember that $s$ is one half the perimeter of the triangle.

Example. — Find the area of the triangle whose sides are 13, 14, and 15.

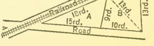
**Solution.** 1. Let $a = 13$, $b = 14$, and $c = 15$.

2. $\therefore s = \frac{1}{2}(13 + 14 + 15) = 21$.

3. $\therefore$ area $= \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2 \times 4 \times 7 \times 3 \times 2}$

4. $= 3 \times 7 \times 2 \times 2 = 84$.

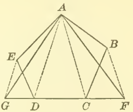
**Ex. 42.** The sides of the lots $A$ and $B$ in the adjoining figure have the lengths indicated. Find the area of each of the lots.




**Note.** — Supplementary Exercises 6 to 30, p. 295, can be studied now.

## Proposition VI. Problem

**336.** *Construct a triangle equal to a given polygon.*




**Given** polygon $ABCDE$.

**Required** to construct a $\triangle = ABCDE$.

I. *Change $ABCDE$ into an equal quadrilateral.*

**Construction.** 1. Draw diagonal $AC$, cutting off $\triangle ABC$.

2. Draw $BF \parallel AC$, meeting $DC$ extended at $F$. Draw $AF$.

**Statement.** $AFDE = ABCDE$.

**Proof.** 1. $\triangle ABC$ and $\triangle ACF$ have the same base, $AC$, and equal altitudes, — the distance between the $\parallel$s $AC$ and $BF$.

2. $\therefore \triangle ABC = \triangle ACF$. Why?

3. $AFDE = ACDE + \triangle ACF$;
and $ABCDE = ACDE + \triangle ABC$. Ax. 3, § 51

4. $\therefore AFDE = ABCDE$. Ax. 7, § 51

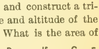
II. *Change $AFDE$ into an equal triangle.*

**Construction.** 1. Draw $AD$; draw $GE \parallel AD$, meeting $FD$ extended at $G$; draw $AG$.

**Statement.** $\triangle AFG =$ quadrilateral $AFDE$. Prove it.

**Ex. 43.** (*a*) Make a reasonably large pentagon, and construct a triangle equal to the pentagon. (*b*) Measure the base and altitude of the triangle, and compute the area of the triangle. (*c*) What is the area of the pentagon?

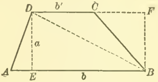
**Ex. 44.** Determine the area of the trapezoid $ABCD$ whose dimensions are indicated in the adjoining figure.




*Suggestions.* — Area of $\triangle ABD =$ ? Area of $\triangle BCD =$ ?

## Proposition VII. Theorem

**337.** *The area of a trapezoid equals one half its altitude multiplied by the sum of its bases.*




**Hypothesis.** Trapezoid $ABCD$ has its altitude $DE = a$, its base $AB = b$, and its base $CD = b'$.

**Conclusion.** Area $ABCD = \frac{1}{2}a(b + b')$.

**Proof.** 1. Draw $BD$ and altitude $BF$ of $\triangle DBC$.

Complete the proof. See Ex. 44.

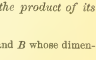
**338. Cor.** *The area of a trapezoid equals the product of its altitude and its median.* (Recall § 153.)

**Ex. 45.** Determine the area of the trapezoids $A$ and $B$ whose dimensions are given in the table below:

| | Altitude | Lower Base | Upper Base |
|---|---|---|---|
| Trapezoid A | 10 in. | 20 in. | 9 in. |
| Trapezoid B | 15 ft. | 30 ft. | 20 ft. |

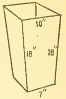
**Ex. 46.** Find the lower base of a trapezoid whose area is 675 sq. ft., upper base 35 ft., and altitude 15 ft.

**Ex. 47.** The non-parallel sides, $AB$ and $CD$, of a trapezoid are each 25 in., and the sides $AD$ and $BC$ are 33 in. and 19 in., respectively. Find the area of the trapezoid.



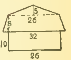

*Suggestions.* — Draw through $B$ a $\parallel$ to $CD$, and a $\perp$ to $AD$.

**Ex. 48.** Prove that the straight line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equal trapezoids.

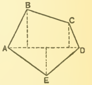



**Ex. 49.** How many square feet of wood will be required for 100 waste-paper boxes like the one pictured in the adjoining figure, — allowing 15% extra for wood wasted in cutting?

**Note.** — Assume that each side is an isosceles trapezoid having the dimensions indicated in the figure.




**Ex. 50.** The adjoining figure represents the end of a barn. If the barn is 35 ft. in length, determine the expense of painting its sides, its end, and its roof at 4¢ per square foot.




**Ex. 51.** The longest diagonal $AD$ of pentagon $ABCDE$ is 44 in., and the perpendiculars to it from $B$, $C$, and $E$ are 24, 16, and 15 in. respectively. If $AB = 25$ in. and $CD = 30$ in., what is the area of the pentagon?

**Ex. 52.** Find the lower base of a trapezoid whose area is 9408 sq. ft., whose upper base is 79 ft., and whose altitude is 96 ft.

**Ex. 53.** Construct a triangle equal to a given trapezoid and having the same altitude as the trapezoid.

**Ex. 54.** Draw through a given point in one side of a parallelogram a straight line, dividing the parallelogram into two equal parts.

**Ex. 55.** Construct a parallelogram equal to a given trapezoid, having the same altitude as the trapezoid.

**Ex. 56.** If $AD$ is the median to side $BC$ of $\triangle ABC$ and $E$ is the mid-point of $AD$, then $\triangle BEC = \frac{1}{2}\triangle ABC$.

**Ex. 57.** If $E$ and $F$ are the mid-points of sides $AB$ and $AC$ respectively of $\triangle ABC$, and $D$ is any point in side $BC$, prove quadrilateral $AEDF = \frac{1}{2}\triangle ABC$.

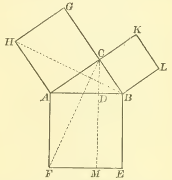
**Ex 58.** If $E$ is any point in side $BC$ of $\square ABCD$, then $\triangle ABE + \triangle ECD = \frac{1}{2}\square ABCD$.

**Ex. 59.** Draw a straight line perpendicular to the bases of a trapezoid which will divide the trapezoid into two equal parts.

**339.** The following Proposition is an alternative demonstration of the Pythagorean Theorem given in Proposition XIV of Book III.

## Proposition VIII. Theorem

**340.** *The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the two legs of the triangle.*




**Hypothesis.** $\angle C$ of $\triangle ABC$ is a right angle.

$ABEF$, $ACGH$, and $BCKL$ are squares.

**Conclusion.** Area $ABEF =$ area $ACGH +$ area $BCKL$.

**Proof.** 1. Draw $CD \perp AB$ and extend it to meet $FE$ at $M$.

2. Draw $BH$ and $CF$.

3. $\triangle ACF \cong \triangle ABH$. (Give the full proof.)

4. $BCG$ is a st. line and parallel to $AH$. § 40.

5. $\therefore \triangle ABH$ and $\square ACGH$ have the same base, $AH$, and equal altitudes, — the distance between the ‖s $BG$ and $AH$.

6. $\therefore$ area $ACGH = 2$ area $\triangle ABH$. Why?

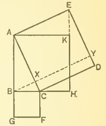
7. Similarly area $ADMF = 2$ area $\triangle ACF$.
(Give the full proof.)

8. $\therefore$ area $ACGH =$ area $ADMF$. Why?
[From steps 6 and 7.]

9. Similarly it can be proved that
area $BCKL =$ area $BDME$.

10. $\therefore$ area $ACGH +$ area $BCKL =$ area $ABEF$.
[From steps 8 and 9.]

**Note.** — Many other proofs of this important theorem can be given. The proof suggested in Ex. 60, which follows, is an extremely suggestive one; the one in Ex. 61 has of course special interest.[1]




**Ex. 60.** Prove the Pythagorean Theorem, using the adjoining figure. (Note. Square $AH$ is "turned in" over $\triangle ABC$.)

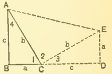
Prove $\square AD = \square BF + \square AH$.

*Suggestions.* — 1. Draw $EK$ and prove $HKE$ is a st. line, by proving $\angle AKE$ is a rt. $\angle$.

2. Prove $\square AH = \square AXYE$, by comparing each with $\triangle ABE$.

3. Prove $\square BF = \square CXYD$.

**Note.** — The Pythagorean Theorem can be proved from figures obtained by "turning in" any of the squares, one at a time, two at a time, or all three of them.




**Ex. 61.** **Garfield's Proof of the Pythagorean Theorem.**

**Hyp.** In $\triangle ABC$, $\angle B = 90°$.

**Con.** $b^2 = a^2 + c^2$.

*Suggestions.* — 1. Extend $BC$ to $D$, making $CD = AB$. Draw $DE \perp BD$ at $D$, making $DE = BC$. Draw $CE$ and $AE$.

2. Prove $ABDE$ is a trapezoid.

3. Express the area of $ABDE$ in terms of $a$ and $c$.

4. Prove $\angle 2 = 90°$, and that $CE = b$.

5. Express the area of $\triangle s$ $ABC$, $CDE$, $ACE$ in terms of $a$, $b$, and $c$.

6. Form an equation based on the fact that the trapezoid consists of the three triangles.

7. Complete the proof algebraically.

**Ex. 62.** Prove $C$, $H$, and $L$ lie in a st. line. (Fig. § 340.) (Draw $CH$ and $CL$, and prove $\angle HCL = 1$ st. $\angle$.)

**Ex. 63.** Prove $AG \parallel BK$.

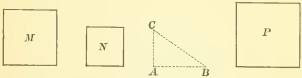
**Ex. 64.** Prove that the sum of the $\perp$s from $H$ and $L$ to $AB$ extended equals $AB$.

*Suggestion.* — Compare $AD$ and $DB$ with $\perp$s.

[1] A number of alternative proofs of the Pythagorean Theorem, and other interesting theorems, are given in Heath's Mathematical Monographs, Numbers 1–4. Published by D. C. Heath & Co., Boston, New York, Chicago. 10¢ each.

## Proposition IX. Problem

**341.** *Construct a square equal to the sum of two given squares.*




**Given** squares $M$ and $N$.

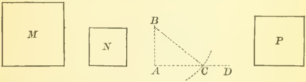
**Required** to construct a square equal to the sum of $M$ and $N$.

**Construction.** 1. Construct $AC \perp AB$, making $AC = n$ and $AB = m$. Draw $BC$.

**Statement.** The square constructed on $BC$ as side $= M + N$.

[Proof to be given by the pupil.]

**342. Cor.** *Construct a square equal to the difference between two given squares.*




**Given** squares $M$ and $N$.

**Required** to construct a square equal to $M - N$.

Construction and proof to be given by the pupil.

[Construction suggested by the figure.]

**Ex. 65.** Construct a square equal to the sum of three given squares.

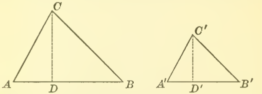
**Ex. 66.** The area of an isosceles right triangle is equal to one fourth the area of the square described upon the hypotenuse.

*Suggestion.*—Compare the right triangle with the square on one leg.

**Ex. 67.** In the figure for § 340, prove that $\triangle\!\!\triangle\ AFH$, $BEL$, and $CGK$ each equals $\triangle ABC$.

## Proposition X. Theorem

**343.** *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



**Hypothesis.** $AB$ and $A'B'$ are homologous sides of similar $\triangle\!\!\triangle$ $ABC$ and $A'B'C'$ respectively.

**Conclusion.** $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

**Proof.** 1. Draw altitudes $CD$ and $C'D'$.

2. $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\frac{1}{2} AB \cdot CD}{\frac{1}{2} \cdot A'B' \cdot C'D'} = \frac{AB \cdot CD}{A'B' \cdot C'D'}.$$ § 333; Ax. 6, § 51

3. $$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \left(\frac{AB}{A'B'}\right) \cdot \left(\frac{CD}{C'D'}\right).$$ An algebraic change

4. But $$\frac{CD}{C'D'} = \frac{AB}{A'B'}.$$ § 282

5. $$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$ Ax. 2, § 51

**Note.** — Since the ratio of two homologous lines of two similar triangles equals the ratio of any two homologous sides, the areas of two similar triangles are to each other as the squares of any two homologous lines.

**Ex. 68.** $\triangle ABC \sim \triangle A'B'C'$ and $AB = 2\,A'B'$.

(*a*) Compare the area of $\triangle ABC$ with the area of $\triangle A'B'C'$.

(*b*) Draw a figure to illustrate the correctness of your result.

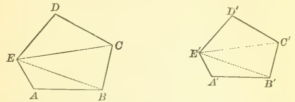
**Ex. 69.** What is the ratio of $\triangle ABC$ to $\triangle A'B'C'$, if they are similar, and:

(*a*) if $AB = 3\,A'B'$? (*b*) if $AB = A'B'$? (*c*) if $AB = \frac{2}{3} A'B'$?

**Note.** — Supplementary Exercises 32 to 35, p. 297, can be studied now.

## Proposition XI. Theorem

**344.** *The areas of two similar polygons are to each other as the squares of any two homologous sides.*




**Hypothesis.** $AB$ and $A'B'$ are homologous sides of similar polygons $AC$ and $A'C'$.

**Conclusion.** $\dfrac{\text{Area of polygon } AC}{\text{Area of polygon } A'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

**Proof.** 1. Draw the diagonals $EB$, $EC$, $E'B'$, and $E'C'$.

2. Then $\triangle ABE \sim \triangle A'B'E'$; $\triangle BCE \sim \triangle B'C'E'$; etc. § 295

3. $\therefore \dfrac{\triangle ABE}{\triangle A'B'E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$. § 343

4. Similarly $\dfrac{\triangle BCE}{\triangle B'C'E'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

$\left[\text{Since } \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}.\right]$

5. Similarly $\dfrac{\triangle CDE}{\triangle C'D'E'} = \dfrac{\overline{CD}^2}{\overline{C'D'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

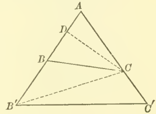
6. $\therefore \dfrac{\triangle ABE}{\triangle A'B'E'} = \dfrac{\triangle BCE}{\triangle B'C'E'} = \dfrac{\triangle CDE}{\triangle C'D'E'}$. Why?

Complete the proof, applying § 296.

**345.** Since the perimeters of two similar polygons have the same ratio as any two homologous sides (§ 297), then *the areas of two similar polygons must have the same ratio as the squares of their perimeters.*

## PROPOSITION XII. THEOREM

**346.** *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.*



**Hypothesis.** $\triangle ABC$ and $\triangle AB'C'$ have $\angle A$ common.

**Conclusion.** $$\frac{\triangle ABC}{\triangle AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

**Proof.** 1. Draw $B'C$; also draw $CD \perp AB'$.

2. $\triangle ABC$ and $\triangle AB'C$ have the common altitude $CD$.

3. $$\therefore \frac{\triangle ABC}{\triangle AB'C} = \frac{AB}{AB'}.$$ Cor. 3, § 334

4. $\triangle AB'C$ and $\triangle AB'C'$ have as common altitude the $\perp$ from $B'$ to $AC'$.

5. $$\therefore \frac{\triangle AB'C}{\triangle AB'C'} = \frac{AC}{AC'}.$$ Why?

6. Multiplying the equations of steps 3 and 5,

$$\frac{\triangle ABC}{\triangle AB'C} \cdot \frac{\triangle AB'C}{\triangle AB'C'} = \frac{AB \times AC}{AB' \times AC'}, \text{ or } \frac{\triangle ABC}{\triangle AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

**Ex. 70.** If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon whose homologous side is 18 in.?

**Ex. 71.** The longest sides of two similar polygons are 18 and 3 in. respectively. How many polygons, each equal to the second, will form a polygon equal to the first?

**Note.** — Supplementary Exercises 36 to 41, p. 298, can be studied now.

### SUPPLEMENTARY TOPICS

Three groups of supplementary material follow. This material appears in some form in most geometries. All of it is interesting and instructive mathematically; none of it is strictly necessary in subsequent parts of geometry.

The teacher should feel free to select the group or groups which best meet the needs of the class.

Group *A*. — Constructions based upon Algebraic Analysis.

This group is especially instructive and interesting.

Group *B*. — Constructions without Formal Analysis.

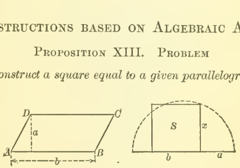
Group *C*. — Miscellaneous Problems.

The first two problems of this group are usually studied. Teachers often omit the remaining ones.

## *A*. Constructions based on Algebraic Analysis

### Proposition XIII. Problem

**347.** *Construct a square equal to a given parallelogram.*




**Given** $\square$ *ABCD* having base $b$ and altitude $a$.

**Required** to construct a square equal to $\square$ *ABCD*.

**Analysis.** 1. Let $x =$ the side of the required square.

2. Then $x^2 =$ the area of the required square, Why?

and $ab =$ the area of the given parallelogram. Why?

3. $\therefore x^2 = ab$. Why?

4. $\therefore a : x = x : b$. § 252

5. $\therefore x$ is the mean proportional between $a$ and $b$, and can be constructed by § 290.

**Construction.** 1. Construct $x$, the mean proportional between $a$ and $b$. § 290

2. On $x$ as side, construct the square $S$.

**Statement.** Square $S = \square ABCD$.

**Proof.** 1. Area of $S = x^2$, and area of $\square ABCD = ab$.

2. $a : x = x : b$, or $x^2 = ab$. Why?

3. $\therefore$ area of $S$ = area of $\square ABCD$.

**Discussion.** The construction is always possible, for the mean proportional between $a$ and $b$ can always be found.

Note 1. — The analysis gives the pupil an idea of how such a construction is discovered. In many cases the proof of the correctness of the resulting construction is rather trivial after such an algebraic analysis, — and in such cases the teacher may decide to omit the proof.

Note 2. — The algebraic solution of such a problem as that proposed in § 347 would duplicate the analysis as far as step 3. Then the 4th step would be: $\therefore x = \sqrt{ab}$. After $x$ had been *computed*, the square would be constructed upon a line of the length determined.

Theoretically the geometric solution is preferable, for $x$ as constructed actually equals the mean proportional between $a$ and $b$, so that the square on side $x$ actually equals the parallelogram; whereas, in the case of the algebraic solution, the value of $x$ is determined only approximately (in most cases) when the square root is found, and hence the square will be only approximately equal to the parallelogram.

**348. Cor.** *Construct a square equal to a given triangle.*

**Analysis.** 1. Let $x$ = the side of the required square, $b$ = the base of the given triangle, and $h$ = the altitude of the triangle.

2. $\therefore x^2 = \frac{1}{2} hb$.

3. $\therefore \frac{1}{2} b : x = x : h$, or $x$ is the mean proportional between $\frac{1}{2} b$ and $h$.

Construction to be given by the pupil.

**Ex. 72.** Construct a square equal to twice a given triangle.

**Ex. 73.** Construct a square equal to twice a given square.

**Ex. 74.** Construct a square which will be twice a given parallelogram.

**Ex. 75.** Construct a square which will be three times a given triangle.

**Ex. 76.** Construct a square which will be two thirds a given rectangle.

**Ex. 77.** Construct a square which will be equal to a given pentagon. (First construct a triangle equal to the pentagon, and then a square equal to the triangle.)

**Ex. 78.** Construct a parallelogram which will equal a given rectangle and have a given segment as base.

**Analysis.** 1. Let $a =$ the altitude and $b =$ the base of the given rectangle, and let $c =$ the given base of the parallelogram. Let $x =$ the required altitude of the parallelogram.

2. Then $cx = ab$. Why?

3. $\therefore c : a = b : x$, or $x$ is the 4th proportional to $c$, $a$, and $b$. Why?

Construction left to the pupil.

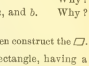
*Suggestion.*—Construct the fourth proportional $x$ and then construct the $\square$.

**Ex. 79.** Construct a rectangle equal to a given rectangle, having a given segment as base. (Analyze as in Ex. 78.)

**Ex. 80.** Construct a triangle equal to a given triangle, having a given segment as base.

*Suggestion.*—Determine the altitude as in Ex. 78, then construct the triangle. How many such triangles can be constructed?

**Ex. 81.** Construct a line parallel to the base of a triangle dividing the triangle into two equal parts.




**Analysis.** 1. Assume $B'C'$ to be the required line: let $AB' = x$.

2. $\therefore \dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{2}{1}$; and $\dfrac{\triangle ABC}{\triangle AB'C'} = \dfrac{\overline{AB}^2}{x^2}$. § 343

3. $\therefore \dfrac{\overline{AB}^2}{x^2} = \dfrac{2}{1}$.

Complete the analysis and then make the construction. That is, determine $x$ first and then draw $B'C'$ at the distance $x$ from $A$ on $AB$.

**Ex. 82.** Construct a rectangle having a given base and equal to $\frac{2}{3}$ a given square. (Analyze as in Ex. 78.)

**Ex. 83.** Construct a triangle having a given base and equal to a given parallelogram.

**Ex. 84.** Construct a parallelogram having a given altitude and equal to a given triangle.

**Ex. 85.** Construct a parallelogram having a given altitude and equal to a given square.

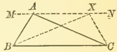
**Ex. 86.** Construct a parallelogram having a given altitude and equal to a given trapezoid.

**Ex. 87.**—Construct a triangle having a given altitude and equal to a given trapezoid.

**Ex. 88.** Construct a square equal to a given trapezoid.

## *B.* Construction without Formal Analysis

**349.** Clearly, if $MN$ is a line parallel to $BC$ through $A$, and $X$ is any point on $MN$, then $\triangle XBC = \triangle ABC$.




In fact, it is clear that:

*The locus of the vertex of a triangle equal to $\triangle ABC$ and having the base $BC$ is a pair of lines parallel to $BC$ at the distance of $A$ from $BC$.*

This fact aids in making numerous constructions.

**Ex. 89.** (*a*) Construct a triangle equal to a given triangle, having the same base $BC$ but having the $\angle XBC = 60°$.

(*b*) Make a similar construction if $\angle XBC = 45°$.

(*c*) Make a similar construction if $\angle XBC = 30°$.

**Ex. 90.** Construct a $\triangle XBC$ equal to a given $\triangle ABC$, having the same base $BC$ and side $XB$ equal to a given segment $d$.

**Ex. 91.** Construct a $\triangle XBC$ equal to a given $\triangle ABC$, and having the median from $X$ to $BC$ equal to a given segment $m$.

**Ex. 92.** Construct a parallelogram $XBCY$ equal to a given $\square ABCD$, having the same base $BC$:

(*a*) having $\angle XBC =$ a given angle;

(*b*) having side $XB =$ a given segment;

(*c*) having diagonal $YB =$ a given segment.

**Ex. 93.** Construct a triangle equal to a given triangle and having two of its sides equal to given segments $m$ and $n$.

*Suggestion.* — Select $m$ as base, and determine the altitude to $m$ as in Ex. 78. Continue as in § 349.

**Ex. 94.** Construct a triangle equal to a given triangle and having one side equal to a given segment $m$, and one angle adjacent to that side equal to a given angle, $\angle T$.

**Ex. 95.** Construct a triangle equal to a given square, having given its base and an angle adjacent to the base.

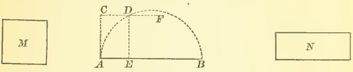
**Ex. 96.** Construct a triangle equal to a given square, having given its base and the median to the base.

**Ex. 97.** Construct a rhombus equal to a given parallelogram, having one of its diagonals coincident with a diagonal of the parallelogram.

## *C.* Miscellaneous Supplementary Problems

### Proposition XIV. Problem

**350.** *Construct a rectangle equal to a given square, having the sum of its base and altitude equal to a given segment.*




**Given** square $M$ and segment $AB$.

**Required** to construct a rectangle $= M$, having the sum of its base and altitude $= AB$.

**Construction.** 1. On $AB$ as diameter construct semicircle $ADB$.

2. Draw $AC \perp AB$, making $AC =$ side of $M$.
3. Draw $CF \parallel AB$, intersecting arc $ADB$ at $D$.
4. Draw $DE \perp AB$.
5. Construct $\square N$, having its base $= BE$ and its altitude $= AE$.

**Statement.** Rectangle $N =$ square $M$.

| | | |
|---|---|---|
| **Proof.** 1. | $AE : DE = DE : BE.$ | § 289 |
| 2. | $\therefore \overline{DE}^2 = AE \times BE.$ | Why? |
| 3. | $\therefore$ area of $M =$ area of $N$. | Why? |

**Discussion.** The construction is impossible when the side of the square is more than $\frac{1}{2} AB$. Why?

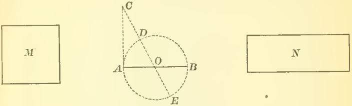
**Note.** — § 350 suggests a geometrical solution of a quadratic of the form $x^2 - tx + m^2 = 0$.

From this equation, $m^2 = x(t - x)$. Clearly, $m$ corresponds to a side of the square, $x(t - x)$ corresponds to the area of the rectangle equal to the square, and $t$ corresponds to the given segment, for $x + (t - x) = t$.

Solve $x^2 - 10x + 16 = 0$ geometrically and check the solution algebraically.

PROPOSITION XV. THEOREM

**351.** *Construct a rectangle equal to a given square, having the difference between its base and altitude equal to a given segment.*



**Given** square $M$ and segment $AB$.

**Required** to construct a rectangle equal to $M$, having the difference between its base and altitude equal to $AB$.

**Construction.** 1. On $AB$ as diameter, construct $\odot\ ADB$.

2. Draw $AC \perp AB$, making $AC =$ a side of $M$.

3. Through the center $O$, draw $CO$, intersecting the $\odot$ at $D$ and $E$.

4. Construct $\square\ N$, having its base $= CE$ and its altitude $= CD$.

**Statement.** $\square\ N$ is the required rectangle.

**Proof.** 1. $CE - CD = DE = AB$;

that is, the base of $N -$ the altitude of $N = AB$.

| | | |
|---|---|---|
| 2. | $AC$ is tangent to the circle. | Why? |
| 3. | $\therefore CE \times CD = \overline{CA}^2$. | § 287 |
| 4. | $\therefore$ area of $N =$ area of $M$. | Why? |

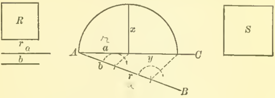
**Discussion.** The construction is always possible, since a secant can always be drawn through the center of the circle from an exterior point.

**Note.** — § 351 suggests a geometrical solution of a quadratic of the form $x^2 + tx - m^2 = 0$, for this equation may be written, $m^2 = x(t + x)$.

Solve the equation $x^2 + 8x - 9 = 0$ geometrically, and check the solution algebraically.

PROPOSITION XVI. PROBLEM

**352.** *Construct a square having a given ratio to a given square.*



**Given** square $R$ and the segments $a$ and $b$.

**Required** to construct a square $S$ such that $S : R = a : b$.

**Analysis.** 1. Let $x =$ one side of the required square.

2. $\therefore x^2 : r^2 = a : b.$

3. $\therefore bx^2 = ar^2$, or $x^2 = \left(\frac{ar}{b}\right) \times r.$ Algebra

4. $\therefore r : x = x : \left(\frac{ar}{b}\right)$, or $x$ is the mean proportional between $r$ and $\frac{ar}{b}$. Why?

5. Let $y = \frac{ar}{b}$, or $by = ar$.

6. $\therefore b : a = r : y$, or $y$ is the fourth proportional to $b$, $a$, and $r$. Why?

**Construction.** 1. Construct $y$ as determined in step 6.

2. Construct $x$ as determined in step 4.

3. Construct square $S$ having $x$ as side.

**Statement.** $S : R = a : b.$

**Proof.** 1. $S : R = x^2 : r^2.$

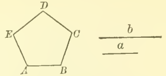
2. But $x^2 = ry$, and $y = \frac{ar}{b}$. Construction

3. $$\frac{S}{R} = \frac{r\left(\frac{ar}{b}\right)}{r^2}.$$ Algebra

4. $$\therefore \frac{S}{R} = \frac{ar^2}{b} \cdot \frac{1}{r^2} = \frac{a}{b}.$$

### Proposition XVII. Problem

**353.** *Construct a polygon similar to a given polygon and having a given ratio to it.*



**Given** polygon $AC$ and segments $a$ and $b$.

**Required** to construct a polygon $A'C'$ similar to $AC$ and such that $\dfrac{\text{polygon } AC}{\text{polygon } A'C'} = \dfrac{a}{b}$.

**Analysis.** 1. Let $x =$ the side homologous to $AB$.

2. Then $\dfrac{\text{polygon } AC}{\text{polygon } A'C'} = \dfrac{\overline{AB}^2}{x^2}$. § 344

3. $\therefore \dfrac{\overline{AB}^2}{x^2} = \dfrac{a}{b}$.

Complete the analysis as in Prop. XVI, thus determining $x$ in terms of $AB$, $a$, and $b$. Then construct upon $x$ as side homologous to $AB$ a polygon similar to polygon $AC$, by § 294. This will be the required polygon.

**Note.** — Notice that § 352 is a special case of § 353, for all squares are similar.

**Ex. 98.** Construct a rectangle similar to a given rectangle and having to it the ratio 2 : 1.

**Ex. 99.** Construct an equilateral triangle equal to a given triangle.

*Suggestion.* — Determine the side $s$ (Ex. 29, Book IV) as in § 348. Recall Ex. 81, Book III.

**Ex. 100.** Construct a triangle equal to the sum of two given triangles.

*Suggestion.* — First construct squares equal to the given triangles.

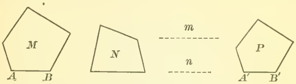
**Ex. 101.** Construct a triangle equal to the difference of two given triangles.

**Ex. 102.** Draw a line parallel to the base of a triangle which will divide the triangle into two parts which will have the ratio 1 : 2.

*Suggestion.* — Analyze as in Ex. 81, Book IV.

### Proposition XVIII. Problem

**354.** *Construct a polygon similar to one of two given polygons and equal to the other.*



**Given** polygons $M$ and $N$.

**Required** to construct a polygon similar to $M$ and equal to $N$.

**Analysis.** 1. Let $m =$ the side of the square equal to $M$, and $n =$ the side of the square equal to $N$. Let $x =$ that side of the required polygon $P$ which is homologous to $AB$.

2. $\therefore \frac{P}{M} = \frac{x^2}{\overline{AB}^2}$. Why?

3. But $\frac{P}{M} = \frac{N}{M} = \frac{n^2}{m^2}$. Since $P = N$.

4. $\therefore \frac{n^2}{m^2} = \frac{x^2}{\overline{AB}^2}$, or $\frac{n}{m} = \frac{x}{AB}$. Why?

5. $\therefore \frac{m}{n} = \frac{AB}{x}$. Why?

6. Hence $x$ is the fourth proportional to $m$, $n$, and $AB$.

**Construction.** 1. Construct the squares equal to $M$ and $N$, thus determining segments $m$ and $n$. See Ex. 77

2. Construct $x$ as determined in step 6.

3. Upon $x$ as side homologous to $AB$, construct a polygon $P$ similar to $M$. § 294

**Statement.** $P \sim M$, and $P = N$.

**Proof.** 1. $P \sim M$. Why?

2. $P : M = x^2 : \overline{AB}^2$. Why?

3. But $x = \frac{AB \times n}{m}$. Construction 2

4. $\therefore P : M = n^2 : m^2$. Substituting in step 2

5. $\therefore P : M = N : M$, or $P = N$.

### Miscellaneous Exercises

**Ex. 103.** A road 60 ft. wide runs from one corner to the opposite corner of a square field measuring 500 ft. on a side, the diagonal of the field running along the center of the road. What is the area of that portion of the field occupied by the road? (Carry out the results to two decimal places.)

**Ex. 104.** What is the length of the side of an equilateral triangle equal to a square whose side is 15 in.?

*Suggestion.* — Recall Ex. 29, Book IV.

**Ex. 105.** From one vertex of a parallelogram, draw lines dividing the parallelogram into three equal parts.

**Ex. 106.** The sides $AB$, $BC$, $CD$, and $DA$ of quadrilateral $ABCD$ are 10, 17, 13, and 20 respectively, and the diagonal $AC$ is 21. Find the area of the quadrilateral.

**Ex. 107.** If diagonals $AC$ and $BD$ of trapezoid $ABCD$ intersect at $E$, then $\triangle AEB = \triangle DEC$. ($BC$ and $AD$ are the bases of $ABCD$.)

*Suggestion.* — Compare $\triangle ABD$ and $\triangle ACD$.

**Ex. 108.** If $X$ is any point in diagonal $AC$ of $\square ABCD$, then

$$\triangle ABX = \triangle AXD.$$

*Suggestion.* — Draw the altitudes from $B$ and $D$ to base $AX$.

**Ex. 109.** If $E$ and $F$ are the mid-points of sides $AB$ and $AC$ of $\triangle ABC$, then $\triangle AEF = \frac{1}{4} \triangle ABC$.

**Ex. 110.** If $E$ is any point within $\square ABCD$, then $\triangle ABE + \triangle CDE$ equals $\frac{1}{2}$ the parallelogram.

*Suggestion.* — Through $E$ draw a line parallel to $AB$.

**Ex. 111.** If $\angle A$ of $\triangle ABC$ is 30°, prove that the area of $\triangle ABC = \frac{1}{4} AB \times AC$.

*Suggestion.* — Draw $BD \perp AC$. Recall Ex. 128, Book I.

**Ex. 112.** Prove that the area of a rhombus is one half the product of its diagonals.

**Ex. 113.** If $E$ is the mid-point of $CD$, one of the non-parallel sides of trapezoid $ABCD$, prove that $ABE = \frac{1}{2} ABCD$.

*Suggestion.* — Through $E$, draw a line parallel to $AB$.

**Ex. 114.** Construct an isosceles triangle equal to a given triangle, having given one side of length $m$.

*Suggestion.* — Use $m$ as the base. Determine the altitude to $m$ as in Ex. 78, Book IV. Then follow § 241.

**Ex. 115.** Draw through a given point in one base of a trapezoid a straight line which will divide the trapezoid into two equal parts.

**Ex. 116.** If the diagonals of a quadrilateral are perpendicular, the sum of the squares on one pair of opposite sides of the quadrilateral equals the sum of the squares on the other pair.

**Note.** — Supplementary Exercises 42 to 46, p. 298, can be studied now.

### Review Questions

1. Define area of a plane figure.

2. Distinguish between congruent, similar, and equal figures.

3. State the rule for determining the area of:

(*a*) a rectangle; (*c*) a triangle;

(*b*) a parallelogram; (*d*) a trapezoid.

4. State the formula for the area of any triangle in terms of its sides $a$, $b$, and $c$, and the number $s$.

What is the number $s$?

5. State the formula for the area of an equilateral triangle in terms of its side $s$.

6. State the corollaries by which the areas of two rectangles are compared:

(*a*) If the rectangles have equal altitudes.

(*b*) If the rectangles have equal bases.

(*c*) When no known relation exists between the altitudes or the bases.

7. State the corresponding corollaries for two parallelograms.

8. State the corresponding corollaries for two triangles

9. State a theorem connecting the areas of a triangle and a parallelogram having equal bases and equal altitudes.

10. State a theorem connecting the areas of two similar polygons.

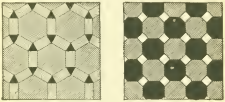
# BOOK V

## REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

**355**. Review the definitions given in § 125, § 128, and § 178.

**356.** A **Regular Polygon** is a polygon which is both equilateral and equiangular.

The figures below illustrate some uses of regular polygons:



TWO LINOLEUM PATTERNS

Notice the regular triangles, hexagons, squares, and octagons.

**Ex. 1.** Prove that the exterior angles at the vertices of a regular polygon are equal.

**Ex. 2.** What is the perimeter of a regular pentagon one of whose sides is 7 in.? of a regular octagon one of whose sides is 6 in.?

**Ex. 3.** In § 154, we have proved that the sum of the angles of any polygon having $n$ sides is $(n-2)$ st. $\angle$s.

How large is each angle of a regular polygon having: ($a$) 3 sides? ($b$) 4 sides? ($c$) 5 sides? ($d$) 6 sides? ($e$) 8 sides? ($f$) 10 sides?

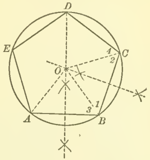
**Ex. 4.** ($a$) Four square tile can be used to cover the space around a point. (Why?)

($b$) In the shape of what other regular polygon can tile be made in order that the surface around a point can be completely covered by using tile of the same shape?

**357.** *Each angle of a regular polygon having n sides is* $\left(\frac{n-2}{n}\right)$ *st.* $\angle$s. (See Ex. 3.)

## Proposition I. Theorem

**358.** *A circle can be circumscribed about any regular polygon.*




**Hypothesis.** $ABCDE$ is a regular polygon.

**Conclusion.** A circle can be circumscribed about $ABCDE$.

**Proof.** 1. A $\odot$ can be constructed through $A$, $B$, and $C$. Let $O$ be its center and $OA$, $OB$, aud $OC$ be radii of it.

2. It can now be proved that this circle passes through $D$ by proving $OD = OA$. (Draw $OD$.)

*Suggestions.*— 1. Compare $\angle ABC$ and $\angle BCD$; $\angle 1$ and $\angle 2$; then $\angle 3$ and $\angle 4$.
2. Prove $\triangle AOB \cong \triangle OCD$, and then $OD = OA$.
3. Hence the $\odot$ passes through $D$.

3. Similarly the circle can be proved to pass through $E$.

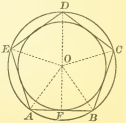
4. Hence a $\odot$ can be circumscribed about $ABCDE$.

**359. Cor. 1.** *A circle can be inscribed in any regular polygon.*

**Proof.** 1. $AB$, $BC$, $CD$, etc. are equal chords of the circle which can be circumscribed about $ABCDE$.

2. Hence these sides are equidistant from $O$. Why?

3. Hence a circle can be drawn tangent to each of the sides of $ABCDE$. § 198



**360.** The **Center** of a regular polygon is the common center of the circumscribed and inscribed circles; as $O$.

The **Radius** of a regular polygon is the distance from its center to any vertex; as $OA$.

The **Apothem** of a regular polygon is the distance from its center to any side; as $OF$.

The **Central Angle** of a regular polygon is the angle between the radii drawn to the ends of any side; as $\angle AOB$.

The **Vertex Angle** of a regular polygon is the angle between two sides of the polygon.

**361. Cor.** 2. *The central angle of a regular n-gon is* $\dfrac{360°}{n}$.

**362. Notation.** The following notation will be employed:

(*a*) $s_4$, $s_6$, or $s_n$ will denote one side of a regular inscribed polygon of 4, 6, or $n$ sides respectively.

(*b*) $a_4$, $a_6$, or $a_n$ will denote the apothem of a regular inscribed polygon of 4, 6, or $n$ sides respectively.

(*c*) $p_4$, $p_6$, or $p_n$ will denote the perimeter of a regular inscribed polygon of 4, 6, or $n$ sides respectively.

(*d*) $k_4$, $k_6$ or $k_n$ will denote the area of a regular inscribed polygon of 4, 6, or $n$ sides respectively.

To denote the corresponding quantities for a regular circumscribed polygon, a capital letter with the appropriate subscript will be employed. Thus,

$S_5 =$ one side of the regular circumscribed pentagon.

$A_5 =$ the apothem of the regular circumscribed pentagon.

$P_5 =$ the perimeter of the regular circumscribed pentagon.

$K_5 =$ the area of the regular circumscribed pentagon.

**Ex. 5.** Find the number of degrees in the central angle and in the vertex angle of a regular polygon of: (*a*) 3 sides; (*b*) 4 sides; (*c*) 5 sides; (*d*) 6 sides; (*e*) 8 sides; (*f*) 10 sides.

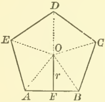
Find also the sum of the central angle and the vertex angle in each case. Do the results suggest any theorem?

**Ex. 6.** Prove that any radius of a regular polygon bisects the angle to whose vertex it is drawn.

**Note.** — Supplementary Exercises 1 to 2, p. 299, can be studied now.

## Proposition II. Theorem

**363.** *The area of a regular polygon is equal to one half the product of its apothem and its perimeter.*



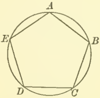

**Hypothesis.** The perimeter of regular polygon $AC$ is $p$ and the apothem $OF$ is $r$.

**Conclusion.** Area of $ABCDE = \frac{1}{2}\,rp$.

**Plan.** From the center $O$ of polygon $AC$, draw the radii $OA$, $OB$, $OC$, etc. forming ⧍ having the common altitude $r$. Determine the area of each triangle and add the results.

## Proposition III. Theorem

**364.** *If a circle be divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon of that number of sides.*

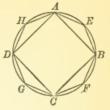



**Hypothesis.** $\widehat{AB} = \widehat{BC} = \widehat{CD} = \widehat{DE} = \widehat{EA}$ in $\odot\ O$.

**Conclusion.** $ABCDE$ is a regular pentagon.

[Proof to be given by the pupil.]

**Note.** — Supplementary Exercise 3, p. 299, can be studied now.



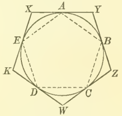

**365. Cor. 1.** *If from the mid-point of each arc subtended by a side of a regular polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

**366. Cor. 2.** *An equilateral polygon inscribed in a circle is regular.*

**Note.** — Supplementary Exercises 4 to 5, p. 299, can be studied now.

## Proposition IV. Theorem

**367.** *If a circle is divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.*




**Hypothesis.** $\odot\ ACD$ is divided into five equal arcs, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, etc. $XY$, $YZ$, etc. are tangent to $\odot\ ACD$ at $A$, $B$, etc., forming pentagon $XYZWK$.

**Conclusion.** $XYZWK$ is a regular pentagon.

*Suggestions.* 1. Draw $AB$, $BC$, $CD$, etc.

2. Prove $\triangle\ AXE$, $AYB$, etc. congruent isosceles $\triangle$.

3. Prove $AX = AY = BY = BZ$, etc.

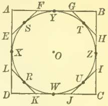
4. Prove $XY = YZ = ZW$, etc.

Recall § 356. Complete the proof.

**Ex. 7.** Prove that any apothem of a regular polygon bisects the side to which it is drawn.

**Ex. 8.** Prove that the diagonals drawn from one vertex of a regular hexagon divides the angle at the vertex into 4 equal angles.

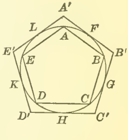
COROLLARIES TO PROPOSITION IV




**368. Cor. 1.** *Tangents drawn to the circle at the mid-points of the arcs included between two consecutive points of contact of a regular circumscribed polygon form, with the sides of the original circumscribed polygon, a regular circumscribed polygon having double the number of sides.*

For the circle is divided into double the number of equal arcs. The theorem follows by § 367.

**369. Cor. 2.** *Tangents drawn to the circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*




If $ABCDE$ is regular, and $F$, $G$, $H$, $K$, and $L$ are the mid-points of arcs $AB$, $BC$, etc., then $\widehat{LF} = \widehat{FG}$, etc. Hence $A'B'C'D'E'$ is regular.

**Note.** — Supplementary Exercise 6, p. 299, can be studied now.

**370. Construction of Regular Polygons** is based upon Propositions III and IV. In order to divide a circle into any number of equal parts, it is sufficient to be able to divide the total angle around the center into that same number of equal angles.

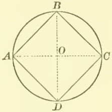
**Ex. 9.** By using your compass, ruler, and *protractor*, draw a regular inscribed pentagon within a circle of radius 2.5 in.

**Ex. 10.** Draw a regular polygon of 9 sides within a circle of 2.5 in. as in Ex. 9.

**371.** It is customary in geometry, however, to use *only the compass and straightedge* in making constructions. It is interesting therefore to inquire what regular polygons can be made by using only these tools.

## Proposition V. Problem

**372.** *Inscribe a square in a given circle.*




**Given** circle $O$.

**Required** to inscribe a square in circle $O$.

**Construction**. 1. Draw $AC$ and $BD$, perpendicular diameters.
2. Draw chords $AB$, $BC$, $CD$, and $AD$.

**Statement**. $ABCD$ is the required square.

**Proof**. 1. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DA}$. Why?
2. $\therefore$ $ABCD$ is an inscribed square. § 364

**373. Cor.** 1. *Regular inscribed polygons of 8, 16, 32, etc. sides can be constructed.* § 365

**Note**. — Hence, by § 372 and § 373, regular inscribed polygons the number of whose sides is a number of the form $2^n$ where $n$ is an integer $\overline{\gtrless} 2$, can be constructed by ruler and compass alone.

Thus, when $n = 2$, $2^n = 4$; when $n = 3$, $2^n = 8$; etc.

**Ex. 11**. Construct within a circle having a 3-in. radius an eight-pointed star like the one which forms the central unit of the adjoining linoleum pattern.

**Ex. 12**. A designer wishes to make a pattern for the octagonal top of a taboret whose longest diagonal is to be 18 in. Make a scale drawing of the octagon, letting 1 in. represent 3 in.

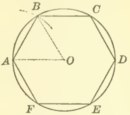
**Ex. 13**. A square is inscribed in a circle of radius 10 in. Compute $s_4$, $a_4$, $p_4$, and $k_4$ to two decimal places. (See § 362.)

**Ex. 14**. A square is circumscribed about a circle of radius 10 in. Compute $S_4$, $P_4$, and $K_4$.

**Note**. — Supplementary Exercises 7 to 14, p. 299, can be studied now.

## PROPOSITION VI. PROBLEM

**374.** *Inscribe a regular hexagon in a circle.*



**Given** circle $O$.

**Required** to inscribe a regular hexagon in circle $O$.

**Analysis.** The central angle of a regular hexagon is 60°.

**Construction.** Draw any radius $OA$. With $A$ as center and $OA$ as radius draw an arc cutting the $\odot$ at $B$.

**Statement.** $\widehat{AB} = \frac{1}{6}$ of the $\odot$, and may be applied 6 times to the circle. The chords of these arcs form the regular inscribed hexagon.

[Proof to be given by the pupil.]

*Suggestions.* — Prove $\angle AOB = 60°$ and that $\widehat{AB} = \frac{1}{6}$ of the circle.

**375. Cor. 1.** *Chords joining the alternate vertices of a regular inscribed hexagon, starting with any vertex, form a regular inscribed triangle.*

**376. Cor. 2.** *Regular inscribed polygons of 12, 24, 48, etc., sides can be constructed.* (§ 365.)

**Note.** — By §§ 374, 375, and 376, regular inscribed polygons the number of whose sides is a number of the form $3 \cdot 2^n$ can be constructed with ruler and compass alone, where $n$ is an integer $\geq 0$.

(What is $3 \cdot 2^n$ when $n = 0$ ? 1 ? 2 ? 3 ?)

**Ex. 15.** Prove that the diagonals joining alternate vertices of a regular hexagon are equal.

**Ex. 16.** Prove that the radii of a regular inscribed hexagon divide it into six congruent equilateral triangles.

**Ex. 17.** Prove that diagonals $AD$, $BE$, and $CF$ of a regular hexagon $ABCDEF$ are the diameters of its circumscribed circle.

**Ex. 18.** Prove that the opposite sides of a regular hexagon are parallel.

**Ex. 19.** Prove that the diagonal $FC$ of regular hexagon $ABCDEF$ is parallel to sides $AB$ and $DE$.

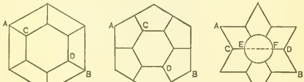
**Ex. 20.** Prove that the diagonal $AD$ of regular hexagon $ABCDEF$ is perpendicular to diagonal $BF$ and bisects it.

**Ex. 21.** A regular hexagon is inscribed in a circle of radius 10 in. Compute the lengths of $s_6$, $a_6$, $p_6$, and $k_6$. (See § 362.)

**Ex. 22.** A regular triangle is inscribed in a circle of radius 10 in. Compute the lengths of $a_3$, $s_3$, $p_3$, and $k_3$.

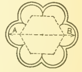
**Ex. 23.** Find the area of a regular hexagon whose apothem is 6 in.

**Ex. 24.** Construct one of the following designs:




$AB = 6''$ $CD = 4''$ | $AB = 6''$ $CD = 3''$ | $AB = 6''$ $CD = 5''$ $EF = 2''$

**Ex. 25.** Construct a pattern for a doily like the one adjoining, making the dimension $AB = 9$ in. and having the radius of the outer arcs $\frac{1}{4}$ in. longer than the radius of the inner arcs of the scallops.




**Note.** — Supplementary Exercises 15 to 31, p. 301, can be studied now.

**377.** A segment is divided by a given point in **Extreme and Mean Ratio** when one part is the mean proportional between the whole segment and the other part.

Thus $C$ divides $AB$ internally in extreme and mean ratio if




$$AB : AC = AC : CB.$$

Notice that in this case the whole segment is to its longer part as the longer part is to the shorter part.

## Proposition VII. Problem

**378.** *Divide a given segment in extreme and mean ratio.*

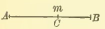



Fig. 1

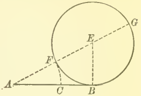



Fig. 2

**Given** segment $AB = m$.

**Required** to divide $AB$ in extreme and mean ratio.

**Analysis.** 1. Let $x = AC$ and $\therefore m - x = CB$.

2. $\therefore m : x = x : (m - x)$.

3. $\therefore x^2 = m(m - x)$.

4. $\therefore x^2 + mx = m^2$.

5. $\therefore x^2 + mx + \left(\frac{m}{2}\right)^2 = m^2 + \left(\frac{m}{2}\right)^2$.

6. $\therefore \left(x + \frac{m}{2}\right)^2 = m^2 + \left(\frac{m}{2}\right)^2$.

7. $\therefore x + \frac{m}{2}$ is the hypotenuse of a right triangle whose base is $m$ and whose altitude is $\frac{m}{2}$.

**Construction.** (Fig. 2.) 1. Draw $EB \perp AB$, making $AB = m$, and $EB = \frac{m}{2}$.

2. Draw $AE$ and on it take $EF = EB = \frac{m}{2}$.

3. On $AB$, take $AC = AF$.

**Statement.** $AB : AC = AC : CB$.

**Note.** — If the equation of step 6 of the Analysis is solved for $x$:

$$x = \pm\sqrt{\frac{5\,m^2}{4}} - \frac{m}{2} = \pm\frac{m}{2}\sqrt{5} - \frac{m}{2}.$$

$$\therefore x = \frac{m}{2}(\sqrt{5} - 1) = \frac{m}{2}(2.236 - 1) = \frac{m}{2}(1.236) = .6\,m.$$

**Proof.*** 1. Complete the circle with center $E$ and radius $EB$, and extend $AE$, cutting the circle at $G$.

2. $AB$ is tangent to $\odot\ BFG$. Why?

3. $\therefore \dfrac{AG}{AB} = \dfrac{AB}{AF}$. § 287

4. $\therefore \dfrac{AG}{AB} = \dfrac{AB}{AC}$. Why?

5. $\therefore \dfrac{AG - AB}{AB} = \dfrac{AB - AC}{AC}$. Why?

6. But $AB = 2\,EB = FG$. Const.

7. $\therefore AG - AB = AG - FG = AF = AC$.

8. $\therefore \dfrac{AC}{AB} = \dfrac{CB}{AC}$.

(Substituting in step 5.)

9. $\therefore \dfrac{AB}{AC} = \dfrac{AC}{CB}$. Why?

**Note 1.** — A point $D$ divides a segment $AB$ externally (see § 305) in extreme and mean ratio if $AB : AD = AD : DB$. This form of division is not used in this text.

**Note 2.** — The Greeks called this method of division of a segment Golden Section. It represented to them the most artistic division of a segment into two parts.

**Ex. 26.** Find $AC$ and $CB$ in § 378 algebraically if $AB = 10$ in.
(Let $AC = x$, and hence $CB = 10 - x$. Continue algebraically.)

**Ex. 27.** (*a*) What is the relation between the area of the inscribed and of the circumscribed squares of a given circle?

(*b*) What is the relation between the perimeter of the inscribed and of the circumscribed squares of a given circle?

**Ex. 28.** Prove that the opposite sides of a regular octagon are parallel.

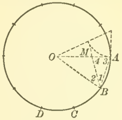
**Ex. 29.** Prove that diagonals $HC$ and $DG$ of regular octagon $ABCDEFGH$ are parallel.

**Ex. 30.** Prove that figure $ACDF$ of regular octagon $ABCDEFGH$ is an isosceles trapezoid.

* This proof may be omitted if desired.

## PROPOSITION VIII. PROBLEM

**379.** *Inscribe a regular decagon in a given circle.*




**Given** $\odot\ ACD$.

**Required** to inscribe a regular decagon in $\odot\ ACD$.

**Construction.** 1. Draw any radius $OA$ and divide it internally at $M$ in extreme and mean ratio so that

$$OA : OM = OM : AM.$$

2. With $A$ as center and $OM$ (the longer segment) as radius, draw an arc cutting the given circle at $B$.

**Statement.** $\overset{\frown}{AB}$ is $\frac{1}{10}$ of the circle, and $AB$ is the side of the regular inscribed decagon.

**Proof.** 1. Draw $OB$ and $BM$.

2. In $\triangle\ OAB$ and $\triangle\ ABM$:

$$\angle A = \angle A;$$

$$OA : AB = AB : AM.$$

[Substituting $AB$ for $OM$ in the proportion of step 1, Construction.]

3. $\therefore \triangle\ OAB \sim \triangle\ ABM.$ § 280

4. $\therefore \angle 1 = \angle\ AOB.$ Why?

5. $\triangle\ OAB$ is isosceles, and hence $\triangle\ ABM$ is isosceles.

6. $\therefore\ OM = AB = BM$, or $\triangle\ OMB$ is isosceles. Why?

7. $\therefore \angle\ AOB = \angle 2.$

8. $\angle 4 = \angle 2 + \angle\ AOB$, or $\angle 4 = 2 \cdot \angle\ AOB.$ Why?

9. $\therefore \angle 3 = 2 \cdot \angle\ AOB.$ Why?

10. $\therefore \angle\ ABO = 2 \cdot \angle\ AOB.$ Why?

11. $\angle\ AOB + \angle 3 + \angle\ ABO = 180°.$ Why?

12. $\therefore 5 \cdot \angle AOB = 180°.$ Why?

13. $\therefore \angle AOB = 36°.$

14. Hence $\widehat{AB} = \frac{1}{10}$ of the circle and $AB$ is one side of the regular inscribed decagon.

**Note.**— This construction is attributed to Pappus.

**380. Cor. 1.** *Chords joining the alternate vertices of a regular inscribed decagon, starting with any vertex, form a regular inscribed pentagon.*

**381. Cor. 2.** *Regular inscribed polygons of 20, 40, 80, etc., sides can be constructed with ruler and compass alone.* Why?

**Note.** — By §§ 379, 380, and 381, regular inscribed polygons the number of whose sides is a number of the form $5 \cdot 2^n$ can be constructed with ruler and compass alone.

(What is $5 \cdot 2^n$ when $n$ is 0? 1? 2? 3? etc.)

**Ex. 31.** In a circle having a 3 in. radius inscribe a regular decagon making all of the constructions.

(Keep the resulting figure for use in later exercises.)

**Ex. 32.** Construct the adjoining figure, having the points $A$ $2\frac{1}{8}$ in. from the center and the points $B$ 3 in. from the center.


(From Ex. 31, obtain the arcs which are $\frac{1}{10}$ of the larger circle.)

**Ex. 33.** Prove that the diagonals of a regular pentagon are equal.

[Construct the pentagon in a circle of radius 3 in., using the arcs obtained in Ex. 31.]

**Ex. 34.** Prove that diagonal $AC$ of regular pentagon $ABCDE$ is parallel to side $DE$. (Circumscribe a ⊙ about the pentagon.)

**Ex. 35.** If $ABCDE$ is a regular inscribed pentagon in circle $O$, prove that a diameter perpendicular to side $DE$ passes through $B$ and is also the perpendicular-bisector of the diagonal $AC$.

**Ex. 36.** If the diagonals $AC$ and $BE$ of a regular inscribed pentagon $ABCDE$ intersect at $F$, prove that $\triangle ABF$ is isosceles.

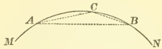
Prove also that $\triangle AEF$ is isosceles.

**Ex. 37.** Construct by ruler and compass alone an angle of 36°; also an angle of 18°.

**Note.** — Supplementary Exercises 32 to 37, p. 301, can be studied now.

## Proposition IX. Problem

**382.** *Inscribe a regular pentadecagon* (15-*gon*) *in a circle.*



**Given** $\odot$ *MN.*

**Required** to inscribe in $\odot$ *MN* a pentadecagon.

**Analysis.** 1. The central $\angle$ of a pentadecagon $= \frac{360°}{15} = 24°$.
2. But $24° = 60° - 36°$.

3. This suggests a combination of the constructions of § 374 and § 379.

**Construction.** 1. Draw chord *AB*, a side of a regular inscribed hexagon, and chord *AC*, a side of a regular inscribed decagon.

2. Draw chord *BC*.

**Statement.** *BC* is one side of the regular inscribed pentadecagon.

**Proof.** 1. $\widehat{BC} = (\frac{1}{6} - \frac{1}{10})$ or $\frac{1}{15}$ of the circle. Const.

**383. Cor.** *Regular polygons of* 30, 60, *etc., sides can be inscribed in a circle by ruler and compass alone.*

**Note.** — Regular polygons, the number of whose sides is a number of the form $15 \cdot 2^n$, where $n$ is an integer, can be constructed with ruler and compass alone. (§§ 382 and 383.)

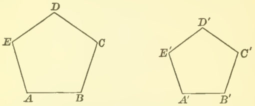
**384.** Combining the results of §§ 373, 376, 381, and 383, it can be said that regular polygons of $2^n$, $3 \cdot 2^n$, $5 \cdot 2^n$, and $15 \cdot 2^n$ sides ($n =$ an integer) can be inscribed in a circle by ruler and compass alone.

**Ex. 38.** How large is the vertex angle of a regular pentadecagon?

**Ex. 39.** What regular polygons having a number of sides less than 100 can be constructed by ruler and compass alone?

## Proposition X. Theorem

**385.** *Regular polygons of the same number of sides are similar.*




**Hypothesis.** $ABCDE$ and $A'B'C'D'E'$ are regular polygons of 5 sides.

**Conclusion.** $ABCDE \sim A'B'C'D'E'$.

**Proof.** 1. The polygons are mutually equiangular.

[Since each $\angle$ of each polygon is $(\frac{540}{5})^\circ$ or $108^\circ$.]

2. Since $AB = BC = CD$, etc., and $A'B' = B'C' = C'D'$, etc.

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.} \qquad \text{Ax. 6, § 51}$$

3. $\therefore$ the polygons have their homologous sides proportional.

4. $\therefore ABCDE \sim A'B'C'D'E'$. Why?

**Ex. 40.** Construct a square having given one of its diagonals.

**Ex. 41.** A square is inscribed in a circle of radius $R$. Prove:

(*a*) $s_4 = R\sqrt{2}$; (*c*) $a_4 = \frac{R}{2}\sqrt{2}$;

(*b*) $p_4 = 4R\sqrt{2}$; (*d*) $k_4 = 2R^2$.

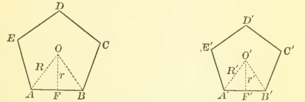
**Ex. 42.** In the figure for § 369, prove that the apothem of the inscribed polygon becomes, when extended, the apothem of the circumscribed polygon.

**Ex. 43.** In the figure for § 367, prove that the radius drawn to any vertex $Y$ is the perpendicular bisector of the side $AB$ of the inscribed polygon.

**Ex. 44.** Prove that the sides of a regular polygon circumscribed about a circle are bisected by the points of tangency.

## Proposition XI. Theorem

**386.** *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*



**Hypothesis.** $P$ and $P'$ are the perimeters, $R$ and $R'$ are the radii, and $r$ and $r'$ are the apothems respectively of the regular polygons $AC$ and $A'C'$ of the same number of sides.

**Conclusion.** $\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$.

**Proof.** 1. Let $O$ and $O'$ be the centers of polygons $AC$ and $A'C'$ respectively.

Draw radii $OA$, $OB$, $O'A'$, and $O'B'$, and apothems $OF$ and $O'F'$.

2. Polygon $AC \sim$ polygon $A'C'$. § 385

3. $\therefore \frac{P}{P'} = \frac{AB}{A'B'}$. § 297

4. $\triangle OAB \sim \triangle O'A'B'$. § 295

5. $\therefore \frac{AB}{A'B'} = \frac{R}{R'}$, and also $\frac{AB}{A'B'} = \frac{r}{r'}$. Why?

6. $\therefore \frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$. Why?

**Ex. 45.** The perimeters of regular inscribed polygons of 6 and 12 sides respectively inscribed in a circle of diameter 2 are approximately 6 in. and 6.21 in. respectively. What are the perimeters of regular inscribed polygons of 6 and 12 sides respectively in a circle of diameter 4? of diameter 7? of diameter 10?

**387. Cor.** *The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.*

1. Let $K$ and $K'$ represent the areas of the polygons $AC$ and $A'C'$ respectively.

Then $$\frac{K}{K'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$ § 344

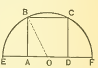
2. $$\frac{K}{K'}=\frac{R^2}{R'^2}=\frac{r^2}{r'^2}.$$ See step 5, § 386

**Ex. 46.** The perimeters of regular polygons of 4 and 8 sides respectively circumscribed about a circle of diameter 2 in. are 8 in. and 6.63 in. respectively. What are the perimeters of regular circumscribed polygons of 4 and 8 sides respectively circumscribed about a circle of diameter 6? of diameter 5?

**Ex. 47.** The area of a regular hexagon inscribed in a circle of radius 3 in. is 23.38 sq. in. What is the area of a regular hexagon inscribed in a circle of radius 6 in.? of one in a circle of radius 1 in.?

**Ex. 48.** The area of a regular octagon circumscribed about a circle of radius 1 in. is 1.656 sq. in. What is the area of a regular octagon circumscribed about a circle of radius 2 in.? of radius 5 in.?

**Ex. 49.** Prove that the square inscribed in a semicircle is equal to two fifths the square inscribed in the entire circle.




*Suggestions.* — Let $R$ = the radius of the circle. Compute the areas of the two squares.

**Ex. 50.** Prove that the perimeter of any regular inscribed polygon is less than the perimeter of the regular circumscribed polygon of the same number of sides.

**Ex. 51.** In a circle of 2 in. radius, inscribe a square, a regular octagon, and also a regular 16-gon. Prove that $p_4 < p_8 < p_{16}$.

**Ex. 52.** About a circle of 2 in. radius circumscribe a square, a regular octagon, and a regular 16-gon. Prove $P_4 > P_8 > P_{16}$.

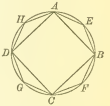
**Ex. 53.** (*a*) Prove that $p_4$, $p_8$, $p_{16}$, etc., are each less than $P_4$.
(*b*) Prove that $P_4$, $P_8$, $P_{16}$, etc., are each greater than $p_4$.

*Suggestions.* — 1. Compare $p_{16}$ with $P_{16}$. See Ex. 50.
2. Compare $P_{16}$ with $P_4$.

## MENSURATION OF A CIRCLE. INFORMAL TREATMENT

**388. Length of a Circle.** We have defined the length of a straight line segment as the ratio of that segment to the unit of linear measure, — another straight line segment. Clearly we cannot define the length of a circle, in that manner, because we cannot lay off the linear unit of measure along a circle. In defining the length of a circle therefore, an entirely new procedure is necessary. The treatment which follows, while informal, involves nevertheless the ideas which underlie the formal treatment of this same topic given in § 401 to § 413 inclusive.

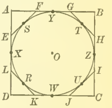
(*a*) In the adjoining circle are inscribed a square and a regular octagon; imagine that the regular inscribed polygons of 16, 32, etc., sides also are drawn.




The perimeters of these polygons have been denoted by $p_4$, $p_8$, $p_{16}$, etc. (§ 362).

We have proved (Ex. 51) that $p_8 > p_4$; that $p_{16} > p_8$; that $p_{32} > p_{16}$; etc. In other words, the perimeters of the regular inscribed polygons increase as the number of sides increases.

(*b*) About the adjoining circle there are circumscribed the regular polygons of 4 and 8 sides. Imagine that those of 16, 32, etc., sides also are drawn.




The perimeters of these polygons have been denoted by $P_4$, $P_8$, $P_{16}$, etc. (§ 362).

We have proved (Ex. 52) that $P_4 > P_8$; that $P_8 > P_{16}$; that $P_{16} > P_{32}$; etc. In other words, the perimeters of the regular circumscribed polygons decrease as the number of sides increases.

(*c*) From the figure it is evident that the successive inscribed polygons come closer and closer to the circle; likewise that the successive circumscribed polygons come closer and closer to the circle.

It is evident also that the length of the circle is greater than the perimeter of any inscribed polygon and that the length of the circle is less thay the perimeter of any circumscribed polygon.

It is natural therefore to regard the successive perimeters of the regular inscribed polygons and also of the regular circumscribed polygons as better and better approximations to the length of the circle.

(*d*) By careful computation it has been found that when the diameter of a circle is 1:

| | |
|---|---|
| $p_4 = 2.82843.$ | $P_4 = 4.$ |
| $p_8 = 3.06147.$ | $P_8 = 3.31371.$ |
| $p_{16} = 3.12145.$ | $P_{16} = 3.18260.$ |
| $p_{32} = 3.13655.$ | $P_{32} = 3.15172.$ |
| $p_{64} = 3.14033.$ | $P_{64} = 3.14412.$ |
| $p_{128} = 3.14128.$ | $P_{128} = 3.14222.$ |
| $p_{256} = 3.14151.$ | $P_{256} = 3.14175.$ |
| $p_{512} = 3.14157.$ | $P_{512} = 3.14163.$ |

Apparently when the diameter of a circle is 1, the length of the circle is approximately 3.1416. If we let $C =$ the length of the circle and $d =$ the length of the diameter, then $C \div d = 3.1416$.

(*e*) By Proposition XI, § 386, the perimeters of regular polygons of the same number of sides have the same ratio as their radii, and hence as their diameters, and also as their apothems.

If we double the diameter of the circle considered in part (*d*), then we shall obtain for the successive perimeters of the inscribed and of the circumscribed polygons exactly double the lengths given in part (*d*). Evidently then the length of a circle of diameter 2 is approximately double that of a circle of diameter 1; that is, $C = 2 \times 3.1416 = 6.2832$. Again, $C \div d = 3.1416$.

Similarly the length of a circle of diameter 5 is approximately $5 \times 3.1416$, or 15.7080. Again $C \div d = 3.1416$.

**389**. The relation derived in parts (*d*) and (*e*) of § 388 is not only apparently true but can be proved to be true. We

shall assume it for the present. It amounts to assuming that the length of a circle bears to the length of its diameter a constant ratio. This fact is proved in § 415.

The Greek letter $\pi$ (pī) is used to denote this constant ratio.

That is, $C \div d = \pi$, or $C = \pi d$.

Two useful approximations of $\pi$ are 3.1416 and $3\frac{1}{7}$.

The length of a circle is called the *Circumference* of the circle.

**Note.** — The determination of the value of $\pi$ and of what sort of number $\pi$ is has been one of the most famous problems of mathematics.

The Egyptians early recognized that $C \div d$ is constant, and obtained for this ratio a value which corresponds to 3.1605.

The Babylonians and Hebrews were content with the much less accurate value, $\pi = 3$. (See I Kings, vii. 23.)

The method employed in this text was introduced by Antiphon (469–399 B.C.), improved by Bryson (a contemporary, probably), and finally carried out arithmetically in a remarkable manner by Archimedes (287–212 B.C.) in a pamphlet on the mensuration of the circle. Antiphon suggested the use of inscribed regular polygons of 4, 8, etc., sides as a means of approximating the length of the circle, and Bryson suggested using at the same time the corresponding circumscribed regular polygons. Archimedes employed inscribed and circumscribed regular polygons having 3, 6, ⋯ 96 sides in his computation, and showed that $\pi > 3\frac{10}{71}$ and $< 3\frac{1}{7}$.

The methods employed by Archimedes remained for a long time the standard procedure in efforts to compute $\pi$. As mathematical skill increased, formulæ for $\pi$ were derived, particularly in trigonometric form, which enabled diligent computers to obtain the value to more and more decimal places.

Vieta (1540–1603) was the first to derive a formula for $\pi$ (not, however, a trigonometric one). He gave for $\pi$ the value 3.141529653. Others carried out the computation to as many as 700 decimal places.

A Holland mathematician, Huygens (1629–1695), at the age of twenty-five, proved some theorems which made it possible to improve greatly on the methods of Archimedes. He was able to obtain from a regular hexagon as accurate a value for $\pi$ as Archimedes obtained from the regular 96-gon.

Mathematicians were particularly interested in determining what kind of number $\pi$ is. In 1766–1767, Lambert proved that it is not rational; *i.e.*, that it cannot be expressed as the quotient of two integers. In 1882, through methods introduced by Hermite in 1873, Lindeman proved that

$\pi$ is a transcendental number; *i.e.*, that it cannot be the root of an ordinary algebraic equation. This was the goal toward which previous efforts had been directed, and thus completely solved a problem to which many of the great mathematicians had given some attention.

**390. Cor. 1.** *The circumference of a circle equals $2\pi r$, where $r$ equals the number of linear units in the radius.*

**391. Cor. 2.** *The circumferences of two circles have the same ratio as their diameters or as their radii.*

**Proof.** Let $r_1$, $d_1$, and $C_1$ be the radius, diameter, and circumference of one circle; and let $r_2$, $d_2$, and $C_2$ be the radius, diameter, and circumference of another circle.

2. Then $C_1 = \pi d_1 = 2\pi r_1$, and $C_2 = \pi d_2 = 2\pi r_2$.

3. $$\therefore \frac{C_1}{C_2} = \frac{\pi d_1}{\pi d_2} = \frac{2\pi r_1}{2\pi r_2}, \text{ or } \frac{C_1}{C_2} = \frac{d_1}{d_2} = \frac{r_1}{r_2}.$$

**Note.** — Remember that this proof is based on an *informal treatment*. For the customary formal treatment of this theorem, read, if it seems desirable, § 414.

**Ex. 54.** Find the circumference of a circle whose diameter is 5 in.; 8 in.; 10 in.

**Ex. 55.** How long is the piece of rubber for the tire of a buggy wheel 4 feet in diameter?

**Ex. 56.** If the diameter of a circle is 48 in., what is the length of an arc of 85°?

**Ex. 57.** How long must the diameter of a circular table be in order to seat 20 people, allowing 30 in. to each person?

(Express the result correct to the nearest inch.)

**Ex. 58.** A fly wheel in an engine room has a diameter of 10 feet. Through how many feet does a point on its outer rim move in a minute if the wheel makes 100 revolutions per second?

**Ex. 59.** (*a*) What is the diameter of a circular race track whose length along its inside edge is one mile?

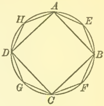
(*b*) If the track is 100 feet wide, determine the distance around it in the middle of the track.

**Ex. 60.** Draw any circle. Construct the circle:

(*a*) Whose circumference is 3 times that of the given circle. See § 391.

(*b*) Whose circumference is $\frac{1}{2}$ that of the given circle.

**392. Area of a Circle.** In the adjoining circle are inscribed a square and a regular octagon. The area of the square is one half the product of its apothem and its perimeter. In symbols (§ 362):



$$k_4 = \tfrac{1}{2} p_4 \times a_4.$$

Similarly the area of the regular octagon is:

$$k_8 = \tfrac{1}{2} p_8 \times a_8.$$

And the area of the regular inscribed 16-gon would be

$$k_{16} = \tfrac{1}{2} p_{16} \times a_{16}.$$

It is evident that the surface within each successive polygon is more nearly equal to the surface within the circle. On the other hand, it is clear that each successive apothem is more nearly equal to the radius and that the length of the polygon is more nearly equal to the length of the circle. (See § 388.)

It is reasonable therefore to conclude that *the area of a circle is one half the product of its radius and its circumference.*

Letting $K$ represent the area of the circle, then

$$K = \frac{1}{2} r \times C.$$

**393. Cor. 1.** Since $C = 2\pi r$, then $K = \frac{1}{2} r \times 2\pi r = \pi r^2$.

**394. Cor. 2.** Since $C = \pi d$, and $r = \dfrac{d}{2}$, then

$$K = \frac{1}{2} \times \frac{d}{2} \times \pi d = \frac{1}{4} \pi d^2.$$

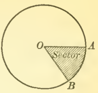
**395. Cor. 3.** *The areas of two circles have the same ratio as the squares of their radii or of their diameters.*

Letting $K_1$ and $K_2$ represent the areas of the circles whose diameters are $d_1$ and $d_2$, and whose radii are $r_1$ and $r_2$ respectively, then

$$\frac{K_1}{K_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{\frac{1}{4}\pi d_1^2}{\frac{1}{4}\pi d_2^2}, \text{ or } \frac{K_1}{K_2} = \frac{r_1^2}{r_2^2} = \frac{d_1^2}{d_2^2}.$$

**Note.** — This theorem was proved by Hippocrates (450–400 B.C.). Look up his history.

**396.** A **Sector of a Circle** is the portion of the interior of a circle which is within a given central angle. The central angle is called the *angle of the sector.*




**397. Cor.** 4. *The area of a sector is one half the product of the radius and the length of the arc intercepted by its angle.*

Let $c =$ the length of the arc and $k =$ the area of the sector of a circle whose area, circumference, and radius are $K$, $C$, and $r$, respectively.

Prove that $\quad k = \frac{1}{2} r \times c.$

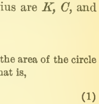
**Proof.** The area of a sector has the same ratio to the area of the circle that the length of its arc has to the circumference; that is,

$$\frac{k}{K} = \frac{c}{C}, \text{ or } k = c \times \frac{K}{C}. \qquad (1)$$

But, since $\quad K = \frac{1}{2} r \times C, \; \frac{K}{C} = \frac{1}{2} r.$

Substituting in (1), $\quad k = c \times \frac{1}{2} r, \text{ or } \frac{1}{2} r \times c.$

**398.** A **Segment of a Circle** is that portion of the interior of a circle which is between a chord of the circle and its subtended arc; as segment $AXB$, indicated by the shaded part of the adjoining figure.




The area of a segment $AXB$ may be determined by subtracting the area of $\triangle AOB$ from the area of sector $OAXB$.

**Ex. 61.** Find the circumference and area of a circle whose diameter is 5 in.; 8 in.; 10 in.

**Ex. 62.** Find the radius and area of a circle whose circumference is $26\pi$ in.; $38\pi$ in.; $15\pi$ in.

**Ex. 63.** Find the radius and circumference of a circle whose area is $64\pi$ sq. in.; $81\pi$ sq. in.; $225\pi$ sq. in.; $289\pi$ sq. in.

**Ex. 64.** Find the side of a square equivalent to a circle whose diameter is 12 in.

**Ex. 65.** The diameters of two circles are 6 and 8 respectively.

(*a*) What is the ratio of their areas?

(*b*) What is the ratio of their circumferences?

**Ex. 66.** The radii of three circles are 3, 4, and 12, respectively. What is the radius of a circle equal to their sum?

**Ex. 67.** Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10. See § 398

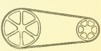
**Ex. 68.** If the radius of a circle is 4, what is the area of a segment whose arc is 120°?

**Ex. 69.** Draw any circle.

(*a*) Construct the circle whose area is four times that of the given circle. § 395

(*b*) Construct the circle whose area is $\frac{1}{4}$ that of the given circle.

**Ex. 70.** Two pulleys in a machine shop are connected by a belt. One has a radius of 9 in. and the other a radius of 1 in. For each revolution of the large pulley how many revolutions will the small pulley make?



**Ex. 71.** What is the area of the ring between two concentric circles whose radii are 8 in. and 10 in. respectively?

**Ex. 72.** A circular grass plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

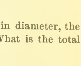
**Ex. 73.** How many tulip bulbs will be required for a circular flower bed 6 feet in diameter, allowing 16 sq. in. to each bulb?

**Ex. 74.** In a steam engine having a piston 20 in. in diameter, the pressure upon the piston is 90 lb. to the square inch. What is the total pressure upon the piston?

**Ex. 75.** A woman had a number of potted plants with which to plant a circular flower bed. She planned to make the bed 4 feet in diameter and found that she used up in that way just one half of her plants. Approximately how large should she make the bed to use up all of her plants?

**Ex. 76.** Prove that the area of the ring included between two concentric circles is equal to the area of a circle whose diameter is that chord of the outer circle which is tangent to the inner.

(To prove area of ring $= \frac{1}{4}\pi AC^2$.)

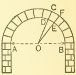


**Ex. 77.** In erecting a hot air furnace for dwellings, certain pipes are installed for carrying the air to the various rooms of the house, and one or more other pipes are put in to convey cold air to the furnace. The cross section area of the cold air supply pipes must equal approximately the sum of the cross section areas of the warm air pipes.

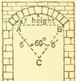
A house is to have four warm air pipes 9 in. in diameter, and three 12 in. in diameter. One cylindrical cold air duct is to be installed. How large, approximately, must its diameter be?

**Ex. 78.** Prove that the area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

**Ex. 79.** In the adjoining *semicircular arch* constructed about center $O$, the distance $AB$ is 10 ft. If the arch is to be constructed of 13 stones of equal size, how long is each of the arcs like arc $DE$?




**Ex. 80.** The adjoining figure represents a *segmental* arch. The method of construction and the dimensions are indicated in the figure. If the arch is made of 11 stones of equal size, what is the length of the arc $XY$? What is the height of the arch?




**Note.** — Supplementary Exercises 38 to 58, p. 302, can be studied now.

## SUPPLEMENTARY TOPICS

Four groups of supplementary topics follow. Each is independent of the others. Teachers should feel free to select the group or groups which appear to meet the needs of the class.

Group *A*. — Inscription of Circles within Regular Polygons and within Circles.

A topic of considerable interest because of its frequent application in artistic design.

Group *B*. — Variables and Limits together with the Formal Treatment of the Mensuration of the Circle and of the Incommensurable Cases.

The treatment of this topic is scientifically correct, but is nevertheless as elementary and pedagogical as the nature of the subject renders possible.

Group *C*. — Symmetry in Plane Figures.

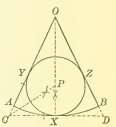
Group *D*. — Maxima and Minima of Plane Figures.

## GROUP A

**399. Inscription of Circles within Regular Polygons and within Circles** is a characteristic feature of art window and other designs.

Many of the necessary constructions are based upon the following illustrative problem or may be discovered by means of an analysis similar to that employed in this problem.

ILLUSTRATIVE PROBLEM. — *Inscribe a circle in a given sector of a circle.*




**Analysis.** Let $\odot\ XYZ$ be tangent to radius $OB$ at $Z$, to $OA$ at $Y$, and to arc $AXB$ at $X$. Let $CD$ be the common tangent to arc $AXB$ and $\odot\ XYZ$ at $X$.

2. Then $\odot\ XYZ$ is inscribed in $\triangle\ OCD$.

3. Hence the center $P$ of $\odot\ XYZ$ lies on the bisectors of $\angle\ COD$ and $\angle\ OCD$.

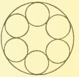
The radius is the distance from $P$ to $X$.

The construction is evident at once.

**Ex. 81.** Construct a circle with radius 2 in.; and within it construct a sector whose angle is 90°.

(*a*) Within this sector inscribe a circle.

(*b*) Compare the area of this circle with the area of the sector itself when the radius of the given circle is $r$ instead of 2.



**Ex. 82.** Construct six equal circles within a circle of radius 2 in., each tangent internally to the given circle and tangent externally to two of the inner circles.

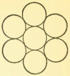
**Ex. 83.** (*a*) If the radius of the given circle in Ex. 82 is $r$, what is the radius of the inscribed circles?

(*b*) Compare the circumference of one of the inscribed circles with the circumference of the given circle.

(*c*) Compare the total area of the inscribed circles with the area of the given circle.

**Ex. 84.** Construct a circle which will be tangent to each of the constructed circles in Ex. 82.

How does the radius of this circle compare with the radius of the six inscribed circles?



**Ex. 85.** Construct six equal circles tangent externally to a circle of radius $\frac{1}{2}$ in. such that each circle is also tangent externally to two of the constructed circles.

**Ex. 86.** If the radius of the given circle in Ex. 85 is $r$:

(*a*) What is the radius of the *escribed* circles?

(*b*) Compare the circumference of the given circle with the circumference of one of the escribed circles.

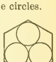
(c) Compare the area of the given circle with the area of one of the escribed circles.

**Ex. 87.** In an equilateral triangle inscribe three equal circles, each tangent to two sides of the triangle and tangent externally to the other two circles.

**Ex. 88.** In a regular hexagon inscribe six equal circles, each tangent to two sides of the hexagon and tangent externally to two of the circles.

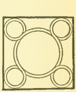
**Ex. 89.** In a regular octagon inscribe eight equal circles, each tangent to two sides of the octagon and also tangent externally to two of the circles.

**Ex. 90.** In a regular hexagon inscribe six equal circles, each tangent to one side of the hexagon and tangent externally to two of the circles.

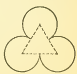


**Ex. 91.** Inscribe in a regular hexagon three equal circles each tangent to two sides of the hexagon and tangent externally to two circles.

**Ex. 92.** In a regular octagon inscribe four equal circles each tangent to two sides of the octagon and also tangent externally to two circles.



**Ex. 93.** The adjoining design appears in a floor pattern in a corridor of the new Congressional Library. Construct such a figure, making a 5-in. square, the radius of the inner ⊙ 1.5 in., and the radius of the concentric ⊙ 1.75 in.



**Ex. 94.** The adjoining curve is a *trefoil.*

(*a*) Construct such a figure based upon an equilateral triangle whose side is 2 in. long.

(*b*) What is the length of the trefoil if the side of the equilateral triangle is $s$ inches?

(*c*) What is the area within the trefoil if the length of the side of the equilateral triangle is $s$ inches?

**Note.** — Supplementary Exercises 59 to 63, p. 304, can be studied now.

## Group B. Mensuration of a Circle

**400.** The formal treatment of the mensuration of a circle involves the use of certain ideas which are fundamental in mathematics.

**401. Variable, Constant,** and **Limit.**

Example 1. — Consider the numbers $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Each number is one half the preceding; while each number is greater than zero, the numbers ultimately become very small. Imagine a literal number $x$ which has these values successively. Then ultimately $x - 0$ becomes less than any small positive number, and *thereafter* remains less than that number. Thus, ultimately, $x - 0$ becomes and remains less than $\frac{1}{1000}$, or $\frac{1}{1,000,000}$, or any other small positive number.

This is clear, since $x$ takes successively the values

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \text{ etc.}$$

Example 2. — Consider the numbers $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16}, \ldots$ Although each number is less than 2, the numbers are constantly increasing.

Imagine a literal number $x$ which has these values successively. Then ultimately $2 - x$ becomes and remains less than any small positive number; thus $2 - x$ becomes and remains less than $\frac{1}{10000}$.

(Write down enough of the successive values of $x$ to make certain of this last statement.)

A **Variable** is a number which assumes different values during a particular discussion. Thus, in Example 1, $x$ is a variable; it is a *decreasing variable*. In Example 2, $x$ is an *increasing variable.*

**Note.** — Variables do not either always increase or always decrease. Thus, the variable which takes the values $1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}$, etc., alternately decreases and increases.

A **Constant** is a number which has a fixed value throughout a particular discussion. Thus, in Example 2, 2 is constant; in Example 1, 0 is constant.

A **Limit of a Variable** is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number.

We say that a variable approaches its limit.

Not every variable has a limit.

**402. Axiom of Limits.** *If an increasing variable is always less than some constant, then it approaches a limit which is less than or equal to that constant. If a decreasing variable is always greater than some constant, then it approaches a limit which is greater than or equal to that constant.*

We may represent the foregoing definitions and axiom geometrically as follows:


Let the distance from $A$ to $X$ as $X$ moves toward $B$ represent a variable $x$. Let it be agreed that $X$ never passes beyond $B$. Then $AX$ must approach a limit such as $AC$ which is less than or equal to $AB$. (In the figure, $AC$ is made less than $AB$.) This means that ultimately point $X$ comes and remains as close to $C$ as we please, possibly even coinciding with $C$.

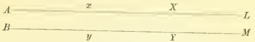
**403. Two Limits Theorems.**

(*a*) *If a variable $x$ approaches a finite limit $l$, then $cx$, where $c$ is a constant, approaches the limit $cl$.*

For $cl - cx = c(l - x)$. As $x$ approaches the limit $l$, the numerical value of $l - x$ becomes and remains less than any small positive number. Hence the numerical value of $cl - cx$ becomes and remains less than any small positive number. Therefore $cx$ approaches the limit $cl$ by definition.

(*b*) *If two variables are constantly equal and each approaches a finite limit, then their limits are equal.*

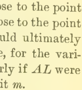
Let $x$ approach the limit $l$ and $y$ approach the limit $m$. If $x$ always equals $y$, then $l$ must equal $m$.




Let the distance $AX$ represent the variable $x$, approaching $AL$. Let the distance $BY$ represent the variable $y$, approaching the limit $BM$.

Point $X$ ultimately must come and remain close to the point $L$; point $Y$ ultimately must come and remain close to the point $M$. If $AL$ were greater than $BM$, then $AX$ would ultimately become greater than $BY$. But this is impossible, for the variable $x$ must always equal the variable $y$. Similarly if $AL$ were less than $BM$. Hence the limit $l$ equals the limit $m$.

**404. Sequences of Regular Polygons.** Let $AB$ be a side of a regular inscribed polygon in a circle of radius $r$; let $AC$ be a side of the regular polygon having double the number of sides of the first; let $AD$ be a side of the regular polygon having double the number of sides of the second; etc.




Such regular inscribed polygons, the number of whose sides is successively doubled, will be called a *sequence of regular inscribed polygons.*

Similarly, we shall have occasion to speak of sequences of regular circumscribed polygons.

**405.** *In a sequence of regular inscribed polygons, the length of the side of the polygon is a decreasing variable which approaches zero as limit, as the number of sides increases indefinitely.*

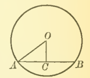
For arc $AC = \frac{1}{2}$ arc $AB$ (see Fig. § 404); arc $AD = \frac{1}{4}$ arc $AB$; etc.

Hence the arcs decrease indefinitely, approaching zero as limit. (See Ex. 1, § 401.)

The chords are less than the corresponding arcs. Hence the chords decrease indefinitely, approaching zero as limit.

Evidently, also in a sequence of regular circumscribed polygons the length of the side is a decreasing variable which approaches zero as limit as the number of sides increases indefinitely.

**406**. *In a sequence of regular inscribed polygons, the length of the apothem is an increasing variable which approaches the radius of the circle as limit.*




If $AB$ is a side of any regular inscribed polygon and $OC$ is the apothem of the polygon, then

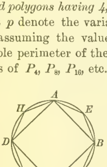
$AO - OC < AC$, or $AO - OC < \frac{1}{2} AB$. (§ 160.)

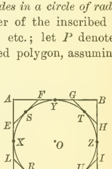
As the number of sides increases indefinitely, $AB$ decreases indefinitely in numerical value. Hence $AO - OC$ must also decrease indefinitely in numerical value. Therefore, $OC$ must approach $AO$ as limit by definition.

## LENGTH OF A CIRCLE

**407**. *Consider the sequences of regular inscribed and circumscribed polygons having 4, 8, 16, etc., sides in a circle of radius r.*

Let $p$ denote the variable perimeter of the inscribed polygon, assuming the values $p_4$, $p_8$, $p_{16}$, etc.; let $P$ denote the variable perimeter of the circumscribed polygon, assuming the values of $P_4$, $P_8$, $P_{16}$, etc.




**408**. *The perimeter p approaches a limit as the number of sides increases indefinitely.*

**Proof.** 1. $p_4 < p_8 < p_{16}$, etc. See Ex. 51

2. $p_4$, $p_8$, $p_{16}$, etc. are all less than $P_4$. See Ex. 53, (a)

3. $\therefore p$ approaches a limit as the number of sides increases indefinitely. Call this limit $l_4$. § 402

**409.** *The perimeter $P$* (see § 407) *approaches a limit as the number of sides increases indefinitely.*

**Proof.** 1. $P_4 > P_8 > P_{16}$, etc. See Ex. 52

2. $P_4$, $P_8$, $P_{16}$, etc. are each greater than $p_4$. See Ex. 53, (*b*)

3. $\therefore$ $P$ approaches a limit as the number of sides increases indefinitely. Call this limit $L_4$. § 402

**410.** *The perimeters $p$ and $P$ of the sequences of regular inscribed and circumscribed polygons described in* § 407 *approach one and the same limit; that is, the limit $l_4$ equals the limit $L_4$.*

**Proof** 1. Let $AB$ = a side of one of the inscribed polygons, and $AD$ and $DB$ halves of two consecutive sides of the circumscribed polygon having the same number of sides. $OD$ and $OA$ are the radii of these polygons. $p$ and $P$ denote their perimeters.

2. The two polygons are similar. § 385

3. $\therefore P : p = OD : OA.$ § 386

4. $\therefore (P - p) : p = (OD - OA) : OA.$ Why?

5. $\therefore P - p = \dfrac{p}{OA}(OD - OA).$ By Algebra

6. $\therefore P - p < \dfrac{P_4}{r} \times AD.$

(Since $p < P_4$; $r = OA$; and $OD - OA < AD$.)

7. Successively double the number of sides, letting the inscribed and the circumscribed polygons always have the same number of sides. The length of each side of the polygons will decrease, approaching the limit zero; in particular, $AD$ will approach 0 as limit.

8. $\therefore$ $\dfrac{P_4}{r} \times AD$ will approach 0 as limit, since $\dfrac{P_4}{r}$ is constant. § 403, (*a*)

9. $P - p$ will approach 0 as limit. Def., § 401

10. $\therefore L_4 = l_4.$

For suppose that $L_4 > l_4$, and that $L_4 - l_4 = m$, a number $> 0$. Ultimately, $P$ differs but little from $L_4$ and $p$ but little from $l_4$; hence $P - p$ ultimately differs but little from $m$. But $P - p$ approaches the limit 0.

Similarly if $L_4 < l_4$.

11. Let $C$ represent the common value of $L_4$ and $l_4$.

Then, as the number of sides increases indefinitely, the perimeters $p$ and $P$ of the regular inscribed and circumscribed polygons respectively approach the limit $C$.

**411.** If now, instead of starting with the sequences of regular polygons having 4, 8, 16, etc. sides, we start with any other sequences of regular inscribed polygons and circumscribed polygons, such as those having 3, 6, 12, etc. sides, it can be proved that the perimeters $p$ and $P$ again approach this same limit $C$ obtained in § 410.

This fact justifies the following definition.

**412. The Length of a Circle** is the limit of the perimeter of any regular inscribed polygon as the number of sides is indefinitely increased.

Remember that the length of a circle is called the circumference of the circle.

**413.** *The perimeter of any regular circumscribed polygon approaches the circumference of the circle as limit if the number of sides is indefinitely increased.* §§ 410, 411, and 412

**Ex. 95.** What is a constant?

**Ex. 96.** What is a variable?

**Ex. 97.** What is the limit of a variable?

**Ex. 98.** If a "sequence" of regular inscribed polygons be formed (§ 404) in a circle:

(*a*) What magnitude is constant?

(*b*) What magnitudes are decreasing variables, and what are their limits?

(*c*) What magnitudes are increasing variables, and what are their limits?

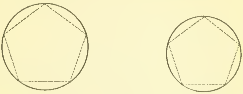
**Ex. 99.** What limit is approached by the variable which assumes the values given in the note at the bottom of page 248?

**Ex. 100.** Suppose a variable assumes the values $1, -1, 1, -1, \cdots$. Does it approach a limit?

**Ex. 101.** What is the limit of $\frac{1}{3^n}$ as $n$ assumes the values 1, 2, 3, 4 …?

## Proposition XII. Theorem

**414.** *The circumferences of two circles have the same ratio as their radii or their diameters.*



**Hypothesis.** $C_1$ and $C_2$ are the circumferences of two circles whose radii are $r_1$ and $r_2$ and whose diameters are $d_1$ and $d_2$ respectively.

**Conclusion.** $\dfrac{C_1}{C_2}=\dfrac{r_1}{r_2}=\dfrac{d_1}{d_2}$.

**Proof.** 1. Inscribe in the circles regular polygons having the same number of sides. Let $p_1$ and $p_2$ be the perimeters of the polygons inscribed in the circles whose radii are $r_1$ and $r_2$ respectively.

2. The polygons are similar. § 385

3. $\dfrac{p_1}{p_2}=\dfrac{r_1}{r_2}$. § 386

4. $\therefore p_1 \times r_2 = p_2 \times r_1$. Why?

5. Let the number of sides of each polygon be successively doubled, the two polygons continuing to have the same number of sides.

6. Then $p_1 \times r_2$ will approach the limit $C_1 \times r_2$ and $p_2 \times r_1$ will approach the limit $C_2 \times r_1$. (§ 411, § 403, (*a*))

7. $\therefore C_1 \times r_2 = C_2 \times r_1$. § 403, (*b*)

8. $\therefore \dfrac{C_1}{C_2}=\dfrac{r_1}{r_2}$. § 252

9. $\therefore \dfrac{C_1}{C_2}=\dfrac{2r_1}{2r_2}=\dfrac{d_1}{d_2}$. Why?

**415. Cor.** Since $\frac{C_1}{C_2}=\frac{d_1}{d_2}$, then $\frac{C_1}{d_1}=\frac{C_2}{d_2}$.

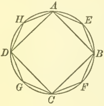
That is, *the ratio of the circumference of a circle to the length of the diameter of the circle is constant for all circles.*

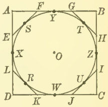
**Note.** — This proves the fact assumed in § 389.

We recall that the constant value $\frac{C}{d}$ is denoted by $\pi$.

**416. Area of a Circle.** The formal treatment of this topic is exactly like that for the length of a circle.

Consider the sequences of regular inscribed and circumscribed polygons in a circle of radius $r$, having 4, 8, 16, etc. sides. (§ 407.)




Let $k$ denote the variable area of the inscribed polygon, and $K$ the variable area of the circumscribed polygon as the number of sides increases.

The following theorems can then be proved:

($a$) The area $k$ approaches a limit as the number of sides increases indefinitely. Call this limit $i_4$.

($b$) The area $K$ approaches a limit as the number of sides increases indefinitely. Call this limit $I_4$.

($c$) The limit $i_4 =$ the limit $I_4$; that is, the areas of the regular inscribed and circumscribed polygons approach the same limit as the number of sides increases indefinitely. Call this area $S$.

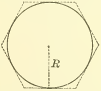
($d$) If any other sequence of regular inscribed or circumscribed polygons of the same circle be formed, the areas of the inscribed and of the circumscribed polygons approach the limit $S$ obtained in the theorem (c).

**417.** The **Area of a Circle** is the limit of the area of a regular inscribed polygon as the number of sides increases indefinitely.

**418.** *The area of any regular circumscribed polygon approaches the area of the circle as limit as the number of sides is increased indefinitely.* This follows at once from theorem (*d*) of § 416.

## Proposition XIII. Theorem

**419.** *The area of a circle is the product of one half its radius and its circumference.*



**Hypothesis.** $r$ is the radius, $C$ is the circumference, and $S$ is the area of the circle.

**Conclusion.** $S = \frac{1}{2} r \times C$.

**Proof.** 1. Circumscribe about the circle any regular polygon. Let $P$ denote its perimeter and $K$ its area.

2. Then, since its apothem is $r$,

$K = \frac{1}{2} r \times P$. Why?

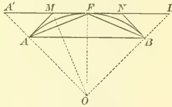
3. Let the number of sides of the circumscribed polygon be indefinitely increased.

| | | |
|---|---|---|
| Then | $K$ will approach the limit $S$; | § 418 |
| | $P$ will approach the limit $C$; | § 411 |
| | $\frac{1}{2} rP$ will approach the limit $\frac{1}{2} rC$. | § 403, (*a*) |
| 4. | $\therefore S = \frac{1}{2} r \cdot C$. | § 403, (*b*) |

**420. The Value of $\pi$.** In § 389, the approximate value 3.1416 for $\pi$ was given. This value was derived from a table of values of perimeters of regular inscribed and circumscribed polygons. The following proposition provides a means of computing such tables of perimeters.

## Proposition XIV. Problem

**421. Given** $p_n$ and $P_n$, the perimeters of the regular inscribed and of the regular circumscribed polygons having $n$ sides; find $p_{2n}$ and $P_{2n}$, the perimeters of the regular inscribed and the regular circumscribed polygons having double the number of sides.




(a) To find $P_{2n}$.

**Solution.** 1. Let $AB$ be one side of the regular inscribed polygon having $n$ sides and $F$ the mid-point of arc $AB$.

2. Draw the tangent to the circle at $F$, meeting $OA$ and $OB$ extended at $A'$ and $B'$ respectively.

Then $A'B'$ is one side of the regular circumscribed polygon having $n$ sides. Also $A'F = \frac{1}{2} A'B'$, and hence $2nA'F = P_n$.

3. Let tangents to the circle at $A$ and $B$ meet $A'B'$ at $M$ and $N$ respectively. Then $MN$ is one side of the regular circumscribed polygon having $2n$ sides; $MF = \frac{1}{2} MN$ and hence $4nMF = P_{2n}$.

4. $$\frac{A'M}{MF} = \frac{OA'}{OF}.$$ § 270

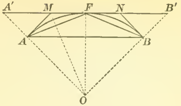
(Since $OM$ bisects $\angle A'OF$.)

5. But $$\frac{P_n}{p_n} = \frac{OA'}{OF}.$$ § 386

6. $$\therefore \frac{P_n}{p_n} = \frac{A'M}{MF}.$$ Why?

7. $$\frac{P_n + p_n}{p_n} = \frac{A'M + MF}{MF}.$$ Why?

8. $$\frac{P_n + p_n}{p_n} = \frac{A'F}{MF}.$$ Why?




9. $$\frac{P_n + p_n}{p_n} = \frac{4\,nA'F}{4\,nMF} = \frac{2\,P_n}{P_{2n}}.$$

[Multiplying num. and denom. of $\frac{A'F}{MF}$ by $4\,n$.]

10. $\therefore P_{2n}(P_n + p_n) = 2\,P_n p_n.$ Why?

11. $$P_{2n} = \frac{2\,P_n p_n}{P_n + p_n}.$$ Algebra

(*b*) To find $p_{2n}$.

1. $\triangle ABF$ and $\triangle AFM$ are isosceles triangles. Why?
2. $\angle ABF = \angle AFM.$ Why?
3. $\therefore \triangle ABF \sim \triangle AFM.$ Why?
4. $AF : AB = MF : AF.$ Why?
5. $\therefore AF^2 = AB \times MF.$ Why?
6. But $AF = \frac{p_{2n}}{2\,n}$; $AB = \frac{p_n}{n}$; and $MF = \frac{P_{2n}}{4\,n}$.

[Steps 2 and 3, part (*a*)]

7. $$\therefore \left(\frac{p_{2n}}{2\,n}\right)^2 = \left(\frac{p_n}{n}\right) \cdot \left(\frac{P_{2n}}{4\,n}\right),$$ Ax. 2, § 51

or $$\frac{p_{2n}{}^2}{4\,n^2} = \frac{p_n P_{2n}}{4\,n^2}.$$ Algebra

8. $\therefore p_{2n} = \sqrt{p_n P_{2n}}.$ Algebra

**Note.** — The formulæ $P_{2n} = \frac{2\,P_n p_n}{P_n + p_n}$ and $p_{2n} = \sqrt{p_n P_{2n}}$ are quite remarkable. If the perimeters of the regular inscribed and regular circumscribed polygons of say 4 sides are known, then the perimeters of the regular circumscribed and regular inscribed polygons of 8 sides can be computed by mere substitution; then those of 16 sides; and so on.

### Proposition XV. Problem

**422.** *Compute an approximate value of $\pi$.*

**Solution.** 1. If the diameter of a circle is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$, and hence the perimeter of the square is $2\sqrt{2}$, or $p_4 = 2.82843$.

2. The side of a circumscribed square is 1, and $P_4 = 4$.

3. $$P_8 = \frac{2 P_4 \times p_4}{P_4 + p_4}. \qquad \S\ 421$$

Hence $$P_8 = \frac{2 \times 4 \times 2.82843}{4 + 2.82843} = 3.31371.$$

4. $$p_8 = \sqrt{p_4 \times P_8}. \qquad \S\ 421$$

Hence $$p_8 = \sqrt{2.82843 \times 3.31371} = 3.06147.$$

5. Similarly $$P_{16} = \frac{2 P_8 \times p_8}{P_8 + p_8} = \frac{2 \times 3.31371 \times 3.06147}{3.31371 + 3.06147} = 3.18260.$$

And $$p_{16} = \sqrt{p_8 \times P_{16}} = \sqrt{3.06147 \times 3.18260} = 3.12145.$$

6. In this manner, we compute the following table:

| No. of Sides | Perimeter of Reg. Circ. Polygon | Perimeter of Reg. Insc. Polygon |
|---|---|---|
| 4 | 4. | 2.82843 |
| 8 | 3.31371 | 3.06147 |
| 16 | 3.18260 | 3.12145 |
| 32 | 3.15172 | 3.13655 |
| 64 | 3.14412 | 3.14033 |
| 128 | 3.14222 | 3.14128 |
| 256 | 3.14175 | 3.14151 |
| 512 | 3.14163 | 3.14157 |

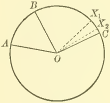
7. The last results show that the circumference of the circle whose diameter is 1 $> 3.14157$ and $< 3\ 14163$.

Hence an approximate value of $\pi$ is 3.1416, correct to the fourth decimal place.

## THE INCOMMENSURABLE CASES *

### Proposition XVI. Theorem

**423**. *In the same circle or in equal circles, central angles have the same ratio as their intercepted arcs.* (When the angles are incommensurable.)




**Hypothesis.** In ⊙ $ABC$, $\angle AOB$ and $\angle BOC$ are two incommensurable central angles intercepting the arcs $AB$ and $BC$ respectively.

**Conclusion.** $$\frac{\angle BOC}{\angle AOB}=\frac{\text{arc } BC}{\text{arc } AB}.$$

**Proof.** 1. Divide $\angle AOB$ into two equal parts and let one of these be applied as unit of measure to $\angle BOC$.

2. Since $\angle AOB$ and $\angle BOC$ are incommensurable, a certain number of angles equal to $\frac{1}{2}\angle AOB$ will equal $\angle BOX_1$, leaving a remainder $\angle X_1OC$ which is less than the unit of measure.

3. $\angle AOB$ and $\angle BOX_1$ are commensurable.

4. $$\frac{\angle BOX_1}{\angle AOB}=\frac{\text{arc } BX_1}{\text{arc } AB}.$$

* **Note.** — There are three incommensurable cases. (§ 423–§ 425.) These propositions complete the proofs of the theorems given in § 213, § 261, and § 327 respectively. If it is desired to read § 423 when studying § 213, then it will be necessary to read also § 401 to § 403 inclusive, which give an introduction to the theory of limits.

5. Take now as unit of measure $\frac{1}{4} \angle AOB$. This measure will be contained an integral number of times in $\angle AOB$ and also in $\angle BOX_1$; further, the unit of measure *may* be contained once in $\angle X_1OC$, leaving a remainder $\angle X_2OC$ which is less than the new unit of measure.

Again $$\frac{\angle BOX_2}{\angle AOB} = \frac{\text{arc } BX_2}{\text{arc } AB}.$$

6. Continue in this manner to decrease indefinitely the unit of measure. The remainder $\angle XOC$, being always less than the unit of measure, will approach the limit $O$.

Using the symbol $\doteq$ to express "approaches the limit,"

$\angle BOX \doteq \angle BOC$, and hence $\dfrac{\angle BOX}{\angle AOB} \doteq \dfrac{\angle BOC}{\angle AOB}$. § 403, (*a*)

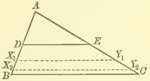
arc $\widehat{BX} \doteq$ arc $\widehat{BC}$, and hence $\dfrac{\text{arc } BX}{\text{arc } AB} \doteq \dfrac{\text{arc } BC}{\text{arc } AB}$. § 403, (*a*)

7. $\dfrac{\angle BOX}{\angle AOB}$ and $\dfrac{\text{arc } BX}{\text{arc } AB}$ are variables which are always equal.

8. $$\therefore \frac{\angle BOC}{\angle AOB} = \frac{\text{arc } BC}{\text{arc } AB}.$$ § 403, (*b*)

### Proposition XVII. Theorem

**424.** *A parallel to one side of a triangle divides the other two sides proportionally, when the segments of one side are incommensurable.*




**Hypothesis.** In $\triangle ABC$, segments $AD$ and $BD$ are incommensurable; $DE \parallel BC$, meeting $AC$ at $E$.

**Conclusion.** $$\frac{BD}{AD}=\frac{CE}{AE}.$$

**Proof.** 1. Divide $AD$ into any number of equal parts (say two), and apply one of these parts to $BD$ as unit of measure.

2. Since $AD$ and $BD$ are incommensurable, a certain number of segments equal to the unit of measure will extend from $D$ to $X_1$, leaving a remainder $X_1B$ which is less than the unit of measure.

3. Draw $X_1Y_1 \parallel BC$, meeting $AC$ at $Y_1$.

Then $$\frac{DX_1}{AD}=\frac{EY_1}{AE}.$$ § 261

[Since $AD$ and $DX_1$ are commensurable.]

4. Take now as unit of measure $\frac{1}{4}$ $AD$. This measure will be contained an integral number of times in $AD$ and also in $DX_1$; further, the unit of measure may be contained once in $X_1B$, leaving a remainder $X_2B$ which is less than the new unit of measure. | Draw $X_2Y_2 \parallel BC$, meeting $AC$ at $Y_2$.

Then $$\frac{DX_2}{AD}=\frac{EY_2}{AE}.$$

5. Continue in this manner to decrease the unit of measure indefinitely. The remainder $XB$, being always less than the unit of measure, will also approach 0 as limit.

Then $DX \doteq DB$, and hence $\frac{DX}{AD} \doteq \frac{DB}{AD}$. § 403, (*a*)

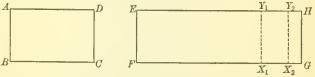
Also $EY \doteq EC$, and hence $\frac{EY}{AE} \doteq \frac{EC}{AE}$. § 403, (*a*)

6. $\frac{DX}{AD}$ and $\frac{EY}{AE}$ are variables which are always equal.

7. $$\therefore \frac{DB}{AD} = \frac{EC}{AE}.$$ § 403, (*b*)

### Proposition XVIII. Theorem

**425.** *Two rectangles having equal altitudes are to each other as their bases, when the bases are incommensurable.*




**Hypothesis.** Rectangles $ABCD$ and $EFGH$ have equal altitudes $AB$ and $EF$, and incommensurable bases $BC$ and $FG$.

**Conclusion.** $$\frac{EFGH}{ABCD} = \frac{FG}{BC}.$$

**Proof.** 1. Divide $BC$ into any number of equal parts (say two), and apply one of these parts to $FG$ as unit of measure.

2. Since $BC$ and $FG$ are incommensurable, a certain number of segments equal to the unit of measure will extend from $F$ to $X_1$, leaving a remainder $X_1G$ which is less than the unit of measure.

3. Draw $X_1Y_1 \perp FG$, meeting $EH$ at $Y_1$. Then rectangles $EFX_1Y_1$ and $ABCD$ have equal altitudes and commensurable bases.

4. $$\therefore \frac{EFX_1Y_1}{ABCD} = \frac{FX_1}{BC}.$$

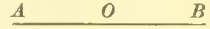
Complete the proof.

*Suggestion.* — Model the proof after that for § 424.

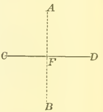
## Group C. Symmetry in Plane Figures

**426. Two points are symmetrical with respect to a third point,** called the **Center of Symmetry,** when the latter bisects the segment which joins them.

Thus, if $O$ is the mid-point of segment $AB$, points $A$ and $B$ are symmetrical with respect to $O$ as center.

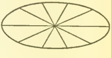



**427. Two points are symmetrical with respect to a straight line,** called the **Axis of Symmetry,** when the latter bisects at right angles the segment which joins them.




Thus, if $CD$ bisects segment $AB$ at right angles, points $A$ and $B$ are symmetrical with respect to $CD$ as an axis.

**428. A figure is symmetrical with respect to a center** when every straight line drawn through the center cuts the figure in two points which are symmetrical with respect to that center.



**429. A figure is symmetrical with respect to an axis** when every straight line perpendicular to the axis cuts the figure in two points which are symmetrical with respect to that axis.

**Ex. 102.** Does a circle have a center of symmetry?
Does it have an axis of symmetry?
Does it have more than one axis of symmetry?

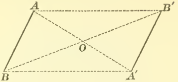
**Ex. 103.** (*a*) Locate upon a sheet of paper a point $O$ and four other points $X$, $Y$, $Z$, and $W$.

(*b*) Construct the points $X'$, $Y'$, $Z'$, and $W'$, which are symmetrical respectively to $X$, $Y$, $Z$, and $W$, with respect to $O$ as center.

**Ex. 104.** (*a*) Draw any straight line $AB$ of indefinite length and upon one side of it locate at random points $X$, $Y$, and $Z$.

(*b*) Construct the points $X'$, $Y'$, and $Z'$, which are respectively symmetrical to $X$, $Y$, and $Z$, with respect to $AB$ as axis.

**430.** THEOREM. *Two segments which are symmetrical with respect to a center are equal and parallel.*




**Hypothesis.** Segments $AB$ and $A'B'$ are symmetrical with respect to center $O$.

**Conclusion.** $AB$ and $A'B'$ are equal and parallel.

**Proof.** 1. Draw lines $AA'$ and $BB'$ intersecting at $O$; draw $AB'$ and $A'B$. § 428

2. $O$ bisects $AA'$ and $BB'$. Prove it.

3. $\therefore AB'A'B$ is a $\square$. Prove it.

4. $\therefore AB$ and $A'B'$ are equal and parallel.

**Ex. 105.** (*a*) Draw a figure something like the adjoining one.


(Let $AB$ and $FG$ be perpendicular to $AG$, and let $BCDEF$ be a curved line.)

(*b*) Construct the figure symmetrical to $ABCDEFG$ with respect to $AG$ as axis.

**Ex. 106.** Prove that two segments which are equal and parallel are symmetrical with respect to a center.

**Ex. 107.** Prove that the bisector of the vertical angle of an isosceles triangle is an axis of symmetry of the triangle.

**Ex. 108.** How many axes of symmetry does an equilateral triangle have?

**Ex. 109.** Prove that the intersection of the diagonals of a parallelogram is the center of symmetry of the parallelogram.

**Ex. 110.** Does a rhombus have a center of symmetry?

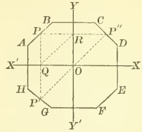
**Ex. 111.** Does the rhombus have an axis of symmetry?

**Ex. 112.** Does a rectangle have an axis of symmetry?

Does it have a second axis of symmetry?

Does it have a center of symmetry?

**431.** THEOREM. *If a figure is symmetrical with respect to each of two perpendicular axes, it is symmetrical with respect to their intersection as center.*




**Hypothesis.** Figure $AE$ is symmetrical with respect to axes $XX'$ and $YY'$. $XX'$ is perpendicular to $YY'$ at $O$.

**Conclusion.** $AE$ is symmetrical with respect to $O$ as center.

**Proof.** 1. From $P'$ any point of $AE$ draw $P'Q \perp$ to $XX'$ at $Q$, and meeting $AE$ again at $P$. Then $PQ = P'Q$.

2. From $P$ draw $PR \perp YY'$ meeting $YY'$ at $R$ and meeting $AE$ again at $P''$. Then $PR = RP''$.

3. Draw $P'P''$.

4. $YY' \parallel PP'$ and bisects $PP''$.

$\therefore$ $YY'$ passes through the mid-point of $P'P''$. Why?

5. Similarly, $XX'$ passes through the mid-point of $P'P''$.

6. Hence $O$, the intersection of $XX'$ and $YY'$, must be the mid-point of $P'P''$.

7. In the same manner, for any other point like $P'$ of $AE$ there is a corresponding point $P''$ of $AE$, such that $P'P''$ passes through $O$ and is bisected by it.

8. $\therefore$ $AE$ is symmetrical with respect to $O$ as center.

**Ex. 113.** Answer for a square the questions proposed in Ex. 112.

**Ex. 114.** Answer for an isosceles trapezoid the questions proposed in Ex. 112.

**Ex. 115.** Answer for a regular hexagon the questions proposed in Ex. 112.

**Ex. 116.** Answer for a regular pentagon the same questions.

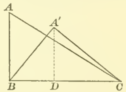
## Group D. Maxima and Minima of Figures

**432.** Two figures are **Isoperimetric** when they have equal perimeters.

**433.** If one geometric magnitude of a number which satisfy certain given conditions has a value greater than that of any of the others, it is called the **Maximum**; if it has a value less than that of any of the others, it is called the **Minimum**.

Thus, of all segments drawn from a given point to a given line the perpendicular is the minimum; again, of all chords of a circle, the diameter is the maximum.

**434.** Theorem. *Of all triangles with two given sides, that in which these sides are perpendicular is the maximum.*



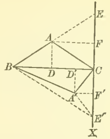

**Hypothesis.** In $\triangle ABC$ and $\triangle A'BC$;
$AB = A'B$; and $AB \perp BC$.

**Conclusion.** Area of $\triangle ABC >$ area of $\triangle A'BC$.

| | | |
|---|---|---|
| **Proof.** 1. | Draw $A'D \perp BC$. | |
| 2. | $A'B > A'D$. | Why? |
| 3. | $\therefore AB > A'D$. | Why? |
| 4. | $\therefore AB \cdot BC > A'D \cdot BC$. [Multiplying both members by $BC$.] | |
| 5. | But area $\triangle ABC = \frac{1}{2} AB \cdot BC$, and area $\triangle A'BC = \frac{1}{2} A'D \cdot BC$. | |
| 6. | $\therefore$ area $\triangle ABC >$ area $A'BC$. | |

**Ex. 117.** Of all parallelograms having two given adjacent sides, that is the maximum in which these sides include a right angle.

**435.** THEOREM. *Of isoperimetric triangles having the same base, that which is isosceles is the maximum.*



**Hypothesis.** $\triangle ABC$ and $\triangle A'BC$ are isoperimetric and have the common base $BC$; $\triangle ABC$ is isosceles.

**Conclusion.** Area of $\triangle ABC >$ area of $\triangle A'BC$.

**Analysis.** We must prove the altitude $AD >$ altitude $A'D'$.

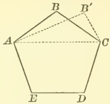
**Proof.** 1. Extend $BA$ to $E$, making $AE = BA$. Draw $EC$.

2. $\angle BCE$ is a rt. $\angle$, for it can be inscribed in a semicircle whose center is $A$ and whose radius is $AB$.

3. Extend $EC$ to $X$. Draw $A'E' = A'C$; draw $BE'$. Construct $A'F'$ and $AF$ both perpendicular to $EE'$.

| | | |
|---|---|---|
| 4. | $BA' + A'E' = BA' + A'C.$ | Why? |
| 5. | $BA + AE = BA + AC.$ | Why? |
| 6. | But $BA' + A'C = BA + AC.$ | Hyp. |
| 7. | $\therefore BA' + A'E' = BA + AE.$ | Why? |
| 8. | $\therefore BA' + A'E' = BE.$ | Why? |
| 9. | But $BA' + A'E' > BE'.$ | Why? |
| 10. | $\therefore BE > BE'.$ | Why? |
| 11. | $\therefore CE > CE'.$ | |
| | [Since $BC \perp EE'$, and $BE > BE'$.] | |
| 12. | $\therefore CF > CF'.$ | |
| | [Since $CF = \frac{1}{2} CE$, and $CF' = \frac{1}{2} CE'$.] | |
| 13. | $\therefore AD > A'D'.$ | |
| | (Prove it.) | |
| 14. | $\therefore$ area of $\triangle ABC >$ area of $\triangle A'BC.$ | Why? |

**436.** THEOREM. *Of isoperimetric polygons having the same number of sides, the maximum is equilateral.*



**Hypothesis.** $ABCDE$ is the maximum of polygons having the given perimeter and the same number of sides as $ABCDE$.

**Conclusion.** $ABCDE$ is equilateral.

**Proof.** 1. Assume that $AB$ and $BC$ are unequal. Draw $AC$.

2. Let $\triangle AB'C$ be the isosceles triangle on base $AC$ having its perimeter equal to that of $\triangle ABC$.

3. Then the area of $\triangle AB'C >$ area of $\triangle ABC$. § 435

4. Then the area of $AB'CDE >$ area of $ABCDE$.

5. But this is impossible for $ABCDE$ is the maximum of all polygons having the same perimeter and the same number of sides as $ABCDE$.

6. $\therefore AB$ and $BC$ cannot be unequal.

7. Similarly $BC = CD = DE$, etc.

8. $\therefore ABCDE$ is equilateral.

**437. Cor.** *Of all isoperimetric triangles, the maximum is equilateral.*

**Ex. 118.** A parallelogram and a rhombus each have a perimeter of 40 in. Which has the greater area?

**Ex. 119.** A man is planning for himself a house. He has a rectangular plan, the dimensions of which are 30 ft. and 20 ft., making the perimeter of the base 100 ft.

Will such a house cover a greater or a less number of square feet than a square house whose perimeter also is 100 ft.?

**438.** THEOREM. *Of isoperimetric equilateral polygons of the same number of sides, the maximum is equiangular.*

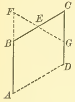



FIG. 1

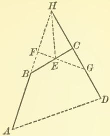



FIG. 2

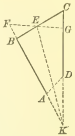



FIG. 3

**Hypothesis.** Consider an equilateral polygon of which $AB$, $BC$, and $CD$ are any three consecutive sides, and whose remaining part is denoted by $P$. Assume that this polygon is the maximum of all equilateral polygons isoperimetric with the given polygon, equal in area to it, and having the same number of sides as it.

**Conclusion.** This polygon is equiangular.

**Plan.** We shall assume that $\angle ABC > \angle BCD$. We shall consider the three cases: Case I. $AB \parallel CD$. Case II. $AB$ meets $CD$ at $H$. Case III. $AB$ meets $CD$ at $K$.

**Proof.** CASE I. (Fig. 1.) 1. Let $E$ be the mid-point of $BC$. Draw $EF$ meeting $AB$ prolonged at $F$, making $EF = BE$. Then $EF$ extended will meet $CD$ at $G$.

2. Then $\triangle BEF \cong \triangle ECG$. Prove it.

3. $\therefore BF = CG$, and $EF = EG$. Why?

4. $\therefore FG = BC$. Why?

5. $\therefore AB + BF + FG + GD = AB + BC + CG + GD$. Why?

6. Hence the polygon composed of $AFGD$ and $P$ has the same perimeter as the given polygon, composed of $ABCD$ and $P$.

7. Also $AFGD$ and $ABCD$ have the same area. Step 2

8. Hence the polygon composed of $AFGD$ and $P$ has the same area as the given polygon, composed of $ABCD$ and $P$.

9. But the given polygon was the maximum of polygons having the given number of sides. Hence the polygon composed of $AFGD$ and $P$ is equal to the maximum of polygons having that number of sides.

10. Hence the polygon composed of $AFGD$ and $P$ is equilateral. § 436

11. But this is impossible since $AF > DG$.

12. Hence $\angle ABC$ cannot be $> \angle BCD$.

CASE II. (Fig. 2.) Assume that $AB$ meets $CD$ at $H$.

1. Let $HE$ bisect $\angle BHC$, meeting $BC$ at $E$.

2. Revolve $\triangle BCH$ on $HE$ as axis until it takes the position of $\triangle FGH$.

3. Then $FG = BC$; $BF = CG$; and $\triangle BEF = \triangle CEG$. Why?

4. $\therefore AB + BF + FG + GD = AB + BC + CG + GD$. Why?

Complete the proof as in steps 6 to 12 inclusive, of Case I.

CASE III. (Fig. 3.) Assume that $AB$ and $CD$ meet at $K$.

1. Let $KE$ bisect $\angle BKC$, meeting $BC$ at $E$.

2. Revolve $\triangle BCK$ on $KE$ as axis until it takes the position of $\triangle FGK$.

3. Then $FG = BC$; $BF = CG$; and $\triangle BEF = \triangle CEG$. Why?

4. $\therefore AB + BF + FG + GD = AB + BC + CG + GD$. Why?

Complete the proof as in steps 6 to 12 inclusive, of Case I.

It follows from Cases I, II, and III that $\angle ABC$ cannot be $> \angle BCD$.

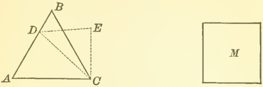
In the same manner, it can be proved $\angle ABC$ cannot be $< \angle BCD$.

Hence $\angle ABC = \angle BCD$.

Since these are any two consecutive angles of the given polygon, then the given polygon must be equiangular.

**439. Cor.** *Of isoperimetric polygons having the same number of sides, the maximum is regular.* (§ 436 and § 437)

**440.** Theorem. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*




**Hypothesis.** $ABC$ is an equilateral triangle, and $M$ is an isoperimetric square.

**Conclusion.** Area of $M >$ area of $\triangle ABC$.

**Proof.** 1. Let $D$ be any point in side $AB$ of $\triangle ABC$.

2. Draw $DC$, and construct upon it as base isosceles $\triangle CDE$ isoperimetric with $\triangle BCD$.

3. Area of $\triangle CDE >$ area of $\triangle BCD$.

4. $\therefore$ area of $ADEC >$ area of $\triangle ABC$.

5. But $ADEC$ and square $M$ are isoperimetric, and hence area of $M >$ area of $ADEC$. § 439

6. $\therefore$ area of $M >$ area of $\triangle ABC$.

In like manner it can be proved that the area of a regular pentagon is greater than that of an isoperimetric square; etc.

**441. Cor.** *The area of a circle is greater than the area of any polygon having an equal perimeter.*

# SUPPLEMENTARY EXERCISES

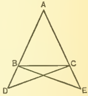
## BOOK I

**Ex. 1.** Two quadrilaterals are congruent if three sides and the two included angles of one are equal respectively to three sides and the two included angles of the other.

*Suggestion.* — Prove by superposition.

**Ex. 2.** Two quadrilaterals are congruent if three angles and the two included sides of one are equal respectively to three angles and the two included sides of the other.

**Ex. 3.** Prove that the base angles of an isosceles triangle are equal, using the following construction.




**Hypothesis.** $AB = AC$.

**Conclusion.** $\angle ABC = \angle ACB$.

**Construction.** Extend $AB$ to $D$. Extend $AC$ to $E$, making $CE = BD$. Draw $DC$ and $BE$.

**Plan.** 1. Prove $\triangle ADC \cong \triangle ABE$ in order to prove $DC = BE$.

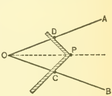
2. Prove $\triangle DBC \cong \triangle BCE$.

3. Prove $\angle DBC = \angle BCE$.

4. Prove $\angle ABC = \angle ACB$.

**Ex. 4.** If $AB$ and $AC$ are two equal chords of the circle whose center is $O$, then the radius $OA$ bisects $\angle BAC$.

**Ex. 5.** Books for carpenters give the following method of bisecting an angle by means of the "square" alone.




Make $OD$ and $OC$ of equal length. Place the square so that $DP = CP$. Then $OP$ bisects $\angle AOB$.

Prove that the method is correct.

**Ex. 6.** Prove that the bisectors of homologous angles of congruent triangles are equal.

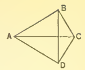
*Suggestions.* — 1. Recall § 66.

2. Remember that the homologous sides and angles of two congruent triangles are equal.

**Ex. 7.** Construct the angle which is double $\angle B$ of Ex. 61, Book I.

**Ex. 8.** Construct the angle which is the sum of $\angle A$ and $\angle B$ of Ex. 61.

**Ex. 9.** Construct an isosceles triangle having its equal sides 3 in. in length and the angle included by them equal to $\angle B$ given in Ex. 61, Book I.

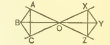




**Ex. 10**. If a diagonal of a quadrilateral $ABCD$ bisects two of its angles, it is perpendicular to the other diagonal and bisects it.

*Suggestions.* — 1. Let $AC$ bisect $\angle A$ and $\angle C$; prove $AC \perp BD$ and $AC$ bisects $BD$.

2. Try to prove $AB = AD$ and $BC = DC$.





**Ex. 11**. In the adjoining figure, if $AO$, $BO$, and $CO$ are extended to $Z$, $Y$, and $X$ respectively, so that $AO = OZ$, $BO = OY$, and $CO = OX$, then $\triangle ABC \cong \triangle ZYX$.

(First prove $AB = ZY$, $BC = XY$, and $AC = XZ$.)

After proving $\triangle ABC \cong \triangle XYZ$, what angle does $\angle BCA$ equal?

**Ex. 12**. Prove that homologous medians of congruent triangles are equal.

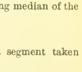
*Suggestion.* — Read the suggestions for Ex. 6, p. 273.

**Ex. 13**. If two triangles have two sides and the median to one of them equal respectively to two sides and the corresponding median of the other, the triangles are congruent.

*Suggestion.* — Read the note following § 77.

**Ex. 14**. Construct the perpendicular-bisector of a segment taken along the lower edge of the paper.

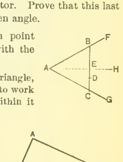
**Ex. 15**. Draw any angle and construct its bisector. Through its vertex construct a line perpendicular to the bisector. Prove that this last line makes equal angles with the sides of the given angle.





**Ex. 16**. Construct a line through a given point within a given acute angle, which will form with the sides of the angle an isosceles triangle.

*Suggestion.* — If $\triangle ABC$ represents the desired triangle, and $AH$ bisects $\angle CAB$, then $CB \perp AH$. Try now to work toward this figure if only $\angle FAG$ and point $D$ within it are given.





**Ex. 17**. Prove Cor. 1 (§ 96) if $\angle 3 = \angle 7$.

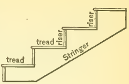
**Ex. 18**. Prove Cor. 3 (§ 98) if $\angle 3 + \angle 5 =$ 1 st. $\angle$.

**Ex. 19**. Prove that $AB \parallel CD$ (Fig. § 98) if $\angle 4 + \angle 7 =$ 1st. $\angle$.

**Ex. 20**. If $AB = CD$ and $\angle 1 = \angle 2$ in the adjoining figure, prove $AB \parallel CD$ and also $AC \parallel BD$.

*Suggestion.* — Recall § 95.

**Ex. 21.** In building stairs two or more *stringers* are required. To make a stringer having a 9-in. *tread* and a 6-in. *riser*, a carpenter uses his square as in the figure below. For each step he places his square so that the 9-in. mark and the 6-in. mark fall along the edge of the board from which he is cutting the stringer.




Will the treads all he parallel? Why?

Will the risers all he parallel? Why?

Will each riser be perpendicular to its tread? Why?


**Ex. 22.** One triangle used by draughtsmen has an angle of 90° and an angle of 60°. Why should it be called a "60–30" triangle?

**Ex. 23.** The other triangle used hy draughtsmen has an angle of 90° and an angle of 45°. How large is the remaining angle?

**Ex. 24.** In a $\triangle ABC$, if $\angle A = 90°$, and $\angle B = \angle C$, how large are $\angle B$ and $\angle C$?

**Ex. 25.** Find the three angles of a triangle if the second is four times the first, and the third is seven times the first. (Algebraic solution.)

**Ex. 26.** Find the three angles of a triangle if the second exceeds the first hy 40°, and the third exceeds the second hy 40°.

**Ex. 27.** The vertical angle of an isosceles triangle is $n$ degrees. Express each of the base angles.

**Ex. 28.** One base angle of an isosceles triangle is $n$ degrees. Express each of the other angles of the triangle.

**Ex. 29.** Determine hy construction the angle $C$ of a $\triangle ABC$ if $\angle A$ and $\angle B$ are the angles given in Ex. 61, Book I.

**Ex. 30.** Prove that two isosceles triangles are congruent when the vertical angle and the base of one are equal respectively to the vertical angle and the base of the other.

*Suggestion.* — Prove the homologous base angles also are equal.

**Ex. 31.** If one acute angle of a right triangle is 35°, how large is the other acute angle?

**Ex. 32.** If perpendiculars he drawn from any point in the hase of an isosceles triangle to the equal sides, they make equal angles with the hase.

*Suggestion.* — The proof is based on § 37 and § 109.

**Ex. 33.** Prove that the altitude drawn to the hypotenuse of a right triangle divides the right angle into two parts which are equal respectively to the acute angles of the right triangle.

**Ex. 34.** If two opposite angles of a quadrilateral are equal and if the diagonal joining the other two angles bisects one of them, then it bisects the other also.

**Ex. 35.** If two triangles have two angles and the bisector of one of these angles equal respectively to two angles and the corresponding bisector of the other, the triangles are congruent.

*Suggestion.* — Recall the note following § 77.

**Ex. 36.** If two triangles have two sides and the altitude drawn to one of them equal respectively to two sides and the corresponding altitude of the other, the triangles are congruent.

*Suggestion.* — Read the note following § 77.

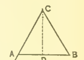
**Ex. 37.** Construct a pattern for the pointed end of a belt, assuming that the belt material is 2 in. wide, and that the point is to project 1 in. beyond the square end of the belt.



**Ex. 38.** Draw any straight line of indefinite length and select two points not in it. Find the point in the line which is equidistant from the two given points.

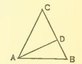
**Ex. 39.** Find a point in one side of a triangle which is equidistant from the other two sides of the triangle.

**Ex. 40.** Prove that either exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one half the vertical angle.




**Ex. 41.** If from the vertex of one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, it makes with the base an angle equal to one half the vertical angle of the triangle.

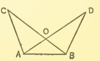
*Suggestion.* — Construct the bisector of $\angle C$.

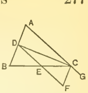



**Ex. 42.** $\triangle ABC$ is an equilateral triangle. $BP$, the bisector of $\angle B$, meets $AC$ at $P$; $CM$, the bisector of exterior angle $ACR$, meets $BP$ extended at $M$. $MN$ is perpendicular to $CR$. Prove $MN = BP$.

**Ex. 43.** If the equal sides of an isosceles triangle be extended beyond the base, the bisectors of the exterior angles so formed form with the base another isosceles triangle.

**Ex. 44.** If $\triangle ABC$ and $\triangle ABD$ are two triangles on the same base and on the same side of it, such that $AC = BD$ and $AD = BC$, and $AD$ and $BC$ intersect at $O$, then $\triangle OAB$ is isosceles.



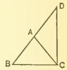



**Ex. 45**. If $CD$ is the bisector of $\angle C$ of $\triangle ABC$, and $DF$ be drawn parallel to $AC$ meeting $BC$ at $E$ and the bisector of the angle exterior to $C$ at $F$, prove $DE = EF$.

*Suggestion.* — Compare $DE$ and $EF$ with $EC$.

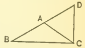
**Ex. 46**. If equiangular triangles be constructed upon the sides of any triangle, the lines drawn from their outer vertices to the opposite vertices of the given triangle are equal.

*Suggestion.* — Recall § 124.




**Ex. 47**. If $AC$ be drawn from the vertex of the right angle to the hypotenuse of right $\triangle BCD$ so as to make $\angle ACD = \angle D$, it bisects the hypotenuse.

*Suggestion.* — Prove $\angle B = \angle ACB$ by § 109 and § 37.




**Ex. 48**. If the angle at the vertex of isosceles $\triangle ABC$ is equal to twice the sum of the equal angles $B$ and $C$, and if $CD$ is perpendicular to $BC$, meeting $BA$ extended at $D$, prove $\triangle ACD$ is equilateral.

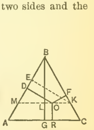
*Suggestion.* — Determine the number of degrees in each angle of $\triangle ABC$.


**Ex. 49**. If the bisectors of the equal angles of an isosceles triangle meet the equal sides at $D$ and $E$ respectively, prove that $DE$ is parallel to the base of the triangle.

*Suggestions.* — 1. Compare $\angle CED + \angle CDE$ with $\angle A + \angle B$ (§ 106). 2. Is $\angle CED = \angle CDE$? 3. Is $\angle CED = \angle A$?

**Ex. 50**. Prove that two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

*Suggestion.* — Prove by superposition. Recall § 132.



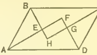

**Ex. 51**. Prove that the sum of the perpendiculars drawn from any point within an equilateral triangle to the sides of the triangle is equal to the altitude of the triangle.

**Prove** $OR + OF + OD = BG$.

*Suggestions.* — 1. Let $KM$ be $\parallel AC$ and $KE \perp AB$. 2. Compare $EK$ and $BL$. 3. Prove $OD + OF = EK$.


**Ex. 52**. An ironing board is supported on each side as shown in the adjoining figure. If $AO = OB$ and $DO = OC$, prove that $AC$ is always parallel to the floor $DB$.

**Ex. 53.** What angle is formed by the bisectors of two consecutive angles of: (*a*) a rectangle? (*b*) an equilateral triangle? (*c*) a parallelogram?




**Ex. 54.** Prove that the bisectors of the interior angles of a parallelogram form a rectangle.

**Ex. 55.** Construct a rhombus whose sides are each 3 in. and whose acute angles are each 45°. Draw and measure its diagonals.

**Ex. 56.** Construct a rhombus, having given one side and one diagonal.

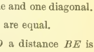
**Ex. 57.** Prove that the two altitudes of a rhombus are equal.

**Ex. 58.** If on the diagonal $BD$ of square $ABCD$ a distance $BE$ is taken equal to $AB$, and if $EF$ is drawn perpendicular to $BD$ meeting $AD$ at $F$, then $AF = EF = ED$.

*Suggestion.*—What kind of angle is angle $EDF$?

**Ex. 59.** If $AD$ and $BD$ are the bisectors of the exterior angles at the ends of the hypotenuse $AB$ of right triangle $ABC$, and $DE$ and $DF$ are perpendicular respectively to $CA$ and $CB$ extended, prove $CEDF$ is a square.

*Suggestion.*—Recall § 143. Prove $DE = DF$, using § 120, I.

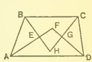



**Ex. 60.** Prove that the bisectors of the angles of a rectangle form a square.

*Suggestions.*—1. Make a plan based upon § 143.
2. To prove $EF = EH$, prove $AF = BH$ and $AE = BE$.

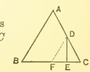
**Ex. 61.** If the non-parallel sides of an isosceles trapezoid are extended until they meet, they form with the base an isosceles triangle.

**Ex. 62.** If the line joining the mid-points of the bases of a trapezoid is perpendicular to the bases, the trapezoid is isosceles.




**Ex. 63.** If the bisectors of the interior angles of a trapezoid do not meet at a point, they form a quadrilateral, two of whose angles are right angles.

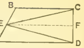
*Suggestion.*—Prove $\angle FEH$ and $\angle FGH$ are right angles.




**Ex. 64.** If $D$ is the mid-point of side $AC$ of isosceles $\triangle ABC$, and $DE$ is perpendicular to base $BC$, then $EC$ is $\frac{1}{4} BC$.

*Suggestion.*—Draw $DF$ parallel to $AB$.

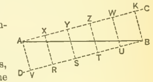
**Ex. 65.** $ABCD$ is a trapezoid whose parallel sides $AD$ and $BC$ are perpendicular to $CD$. If $E$ is the mid-point of $AB$, prove $EC = ED$.




*Suggestion.* — Draw $EF$ parallel to $AD$. Recall § 149.

**Ex. 66.** The following method of dividing a segment into equal segments may be used.

To divide $AB$ into five equal parts.



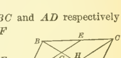

1. Draw $AC$ making with $AB$ any convenient angle.
2. Draw $BD$ parallel to $AC$.
3. Lay off on $AC$ five equal segments, and on $BD$ five other segments of the same length.
4. Connect the points of division as in the figure.

Prove now that $AB$ is divided into five equal segments.

**Ex. 67.** If the base of an isosceles triangle be trisected, the lines joining the points of trisection to the vertex of the triangle are equal.

**Ex. 68.** Prove that the line which joins the mid-points of two sides of a triangle bisects any segment drawn to the third side from the opposite vertex.

**Ex. 69.** If $E$ and $F$ are the mid-points of $BC$ and $AD$ respectively of parallelogram $ABCD$, prove that $AE$ and $CF$ trisect $BD$.

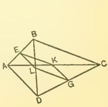



*Suggestion.* — Prove $AE \parallel FC$, by proving $AECF$ is a parallelogram. Then prove that $AE$ bisects $BH$ and $CF$ bisects $GD$.

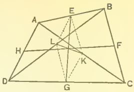
**Ex. 70.** If $E$ and $F$ are the mid-points of sides $AB$ and $AC$ respectively of $\triangle ABC$, and $AD$ is the perpendicular from $A$ to $BC$, prove $\angle EDF = \angle EAF$.

*Suggestion.* — Recall Ex. 175, Book I.

**Ex. 71.** If $E$ and $G$ are the mid-points of $AB$ and $CD$ respectively of quadrilateral $ABCD$, and $K$ and $L$ are the mid-points of diagonals $AC$ and $BD$ respectively, prove that $EKGL$ is a parallelogram.



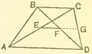

*Suggestion.* — Recall § 151.




**Ex. 72.** Prove that the lines joining the mid-points of the opposite sides of a quadrilateral and the line joining the mid-points of the diagonals of the quadrilateral meet in a point.

*Suggestion.* — Recall Ex. 71.

**Ex. 73.** Prove that the line joining the mid-points of the diagonals of a trapezoid is parallel to the bases and equal to $\frac{1}{2}$ their difference.

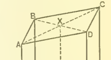



*Suggestions.* — 1. Draw $EG \parallel AD$ meeting $CD$ at $G$.

2. Prove $EG$ passes through point $F$.

3. Compare $EG$ with $AD$ and $FG$ with $BC$.

**Ex. 74.** If the perpendiculars $AE$, $BF$, $CG$, and $DH$ be drawn from the vertices of parallelogram $ABCD$ to any line in its plane not intersecting its surface, prove that $AE + CG = BF + DH$.




*Suggestion.* — See adjoining figure. Apply § 153.

**Ex. 75.** Prove that the sum of any three sides of a quadrilateral is greater than the fourth side.

*Suggestion.* — Draw a diagonal.

**Ex. 76.** Prove that the sum of the lines drawn from any point within a triangle to the vertices is less than the sum of the three sides.

*Suggestion.* — 1. Let $O$ within $\triangle ABC$ be joined to $A$, $B$, and $C$.

2. $OA + OB < AC + BC$. (Ex. 188, Book I.)

3. Similarly express $OB + OC$ and also $OC + OA$.

4. Add these inequalities and divide by 2.

**Ex. 77.** In triangle $ABC$, if $D$ is any point on $AC$ so that $AD = AB$, then $BC > DC$.

*Suggestion.* — Compare $BC + AB$ with $AC$.

**Ex. 78.** Prove that each of the equal sides of an isosceles triangle is greater than one half the base.

**Ex. 79.** If $O$ is any point within triangle $ABC$, then $AO + BO + CO > \frac{1}{2}$ perimeter.

*Suggestions.* — 1. Apply § 159 (*a*) to each side of the triangle.

2. Add the inequalities and divide by 2.

**Ex. 80.** Prove that any side of a triangle is less than one half the perimeter of the triangle.

*Suggestions.* — 1. Apply § 159 (*a*) to one side.

2. Add that side to both members of the inequality.

**Ex. 81.** Prove that the median to any side of a triangle is less than one half the perimeter of the triangle.

*Suggestion.* — The median lies in each of two △s.

Apply § 159 (*a*) to the median in each △, and add.

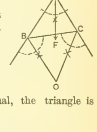
**Ex. 82.** Prove that the median to any side of a triangle is less than one half the sum of the other two sides of the triangle.

*Suggestion.* — 1. Extend the median its own length, through the side of the triangle. Connect the end of the new segment with one of the other vertices of the triangle.

**Ex. 83.** Prove that the median to any side of a triangle is greater than one half the sum of the other two sides diminished by the side to which it is drawn.

**Ex. 84.** Prove that the sum of the medians to the sides of a triangle is greater than one half the perimeter of the triangle.

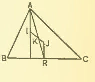
*Suggestion.* — Apply Ex. 83 to each of the medians.




**Ex. 85.** The bisectors of the exterior angles at two vertices, and the bisector of the interior angle at the third vertex of a triangle are concurrent.

*Suggestion.* — The proof is like that for § 169.

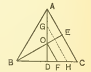
**Ex. 86** If two medians of a triangle are equal, the triangle is isosceles.




**Ex. 87.** Prove that the line joining the orthocenter of a triangle to the circum-center of the triangle passes through the center of gravity (§ 178) of the triangle.

*Suggestions.* — 1. Draw $AR$, and try to prove that $K$ is the center of gravity, by proving that $AK = 2\,KR$.

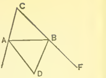
2. Recall § 152, and Ex. 190, Book I.




**Ex. 88.** If $O$ is the point of intersection of the medians $AD$ and $BE$ of equilateral triangle $ABC$, and $OF$ is drawn parallel to $AC$, meeting $BC$ at $F$, prove that $DF$ is $\frac{1}{6}$ $BC$.

*Suggestion.* — Let $G$ be the mid-point of $OA$, and draw $GH \parallel AC$; also imagine a line through $D \parallel AC$. Apply § 147.

**Ex. 89.** If the exterior angles at the vertices $A$ and $B$ of $\triangle ABC$ are bisected by lines which meet at $D$, prove $\angle D = \frac{1}{2} \angle B + \frac{1}{2} \angle A$.



**Proof.** 1. $\angle D = 180 - \angle DAB - \angle ABD$. Why?

2. $\angle DAB = \frac{1}{2}\angle EAB = \frac{1}{2}(\angle C + \angle ABC)$. Why?

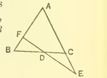
3. Similarly $\angle ABD = ?$

4. $180° = \angle BAC + \angle C + \angle ABC$. Why?

5. Substitute in step 1, and complete the proof.

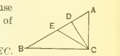
NOTE. — This proof is typical of many that involve numerical relations among angles of a figure. In triangles, the facts in §§ 106, 109, and 110 are used frequently.

**Ex. 90.** Prove that the exterior angle at the base of an isosceles triangle equals the angle between the bisectors of the base angles.



**Ex. 91.** $D$ is any point in the base $BC$ of isosceles triangle $ABC$. The side $AC$ is extended from $C$ to $E$, so that $CE$ equals $CD$, and $DE$ is drawn, meeting $AB$ at $F$. Prove $\angle AFE = 3\angle AEF$.

*Suggestions.* — 1. $\angle AFE$ is exterior to $\triangle BDF$.
2. $\angle B = \angle ACD$, which is exterior to $\triangle CDE$.



**Ex. 92.** If $CD$ is the altitude to the hypotenuse $AB$ of right triangle $ABC$, and $E$ is the mid-point of $AB$, prove $\angle DCE = \angle A - \angle B$.

*Suggestions.* — 1. $\angle DCE$ is the complement of $\angle DEC$. Why?
2. Express $\angle DEC$. 3. Recall Ex. 175, Book I.

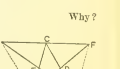
**Ex. 93.** If $CD$ is the altitude to the hypotenuse $AB$ of right triangle $ABC$, and $CE$ is the bisector of $\angle C$, meeting $AB$ at $E$, then $\angle DCE = \frac{1}{2}(\angle A - \angle B)$.

*Suggestions.* — 1. $\angle DCE = \angle ACE - \angle ACD$. Why?
2. $\angle ACE = \frac{1}{2} 90°$. 3. $90° = \angle A + \angle B$. Why?

**Ex. 94.** If $\angle B$ of $\triangle ABC$ is greater than $\angle C$, and $BD$ is drawn to $AC$ making $AD$ equal to $AB$, prove

$$\angle ADB = \tfrac{1}{2}(\angle B + \angle C), \text{ and } \angle CBD = \tfrac{1}{2}(\angle B - \angle C).$$

*Suggestions.* — 1. First apply § 110.
2. $\angle DBC = \angle B - \angle ABD = \angle B - \angle ADB$. Why?

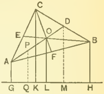


**Ex. 95.** If $D$ and $E$ are the mid-points of sides $BC$ and $AC$ respectively, of $\triangle ABC$, and $AD$ be extended to $F$ and $BE$ to $G$, making $DF = AD$ and $EG = BE$, prove that $GCF$ is a straight line and that $GC = CF$.

*Suggestion.* — Recall Ex. 146, Book I, and § 90.

**Ex. 96.** If the median drawn from any vertex of a triangle is greater than, equal to, or less than one half the opposite side, the angle at the vertex is acute, right, or obtuse respectively. (§ 161.)

**Ex. 97.** The perpendicular from the intersection of the medians of a triangle to any straight line in the plane of the triangle, not intersecting its surface, is equal to one third the sum of the perpendiculars from the vertices of the triangle to the same line. (§ 153.)




**Ex. 98.** Prove that the diagonals of an oblique-angled parallelogram are unequal, the one joining the acute angles being the greater.

**Ex. 99.** Define:

(*a*) parallelogram
(*b*) rectangle
(*c*) square
(*d*) rhombus
(*e*) trapezoid
(*f*) isosceles trapezoid
(*g*) altitude of a parallelogram
(*h*) altitude of a trapezoid
(*i*) median of a trapezoid

**Ex. 100.** What are the important facts known about every parallelogram?

**Ex. 101.** State four theorems by which a quadrilateral can be proved a parallelogram.

**Ex. 102.** State facts known about a rectangle.

**Ex. 103.** State facts known about a square.

**Ex. 104.** State facts known about a trapezoid.

**Ex. 105.** State facts known about an isosceles trapezoid.

**Ex. 106.** State methods for proving two segments equal.

**Ex. 107.** State methods for proving two angles equal.

**Ex. 108.** State methods for proving two lines are parallel.

## BOOK II

**Ex. 1.** If $AB$ is one of the non-parallel sides of a trapezoid circumscribed about a circle whose center is $O$, prove $\angle AOB$ is a right angle.

*Suggestion.* — Recall Ex. 38 (*b*), p. 106.

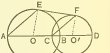
**Ex. 2.** The straight line joining the mid-points of the non-parallel sides of a circumscribed trapezoid is equal to $\frac{1}{4}$ the perimeter of the trapezoid.

*Suggestion.* — Recall Ex. 39, p. 106.

**Ex. 3.** If tangents are drawn to a circle at the extremities of any pair of diameters which are not perpendicular to each other, the figure formed is a rhombus. (Recall Ex. 39, p. 106.)

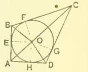
**Ex. 4.** If the angles of a circumscribed quadrilateral are right angles, the figure is a square.

**Ex. 5.** $A$, $B$, $C$, and $D$ are four points in a straight line, $B$ lying between $C$ and $D$; $EF$ is a common tangent to the circles drawn upon $AB$ and $CD$ as diameters. Prove $\angle BAE = \angle DCF$.

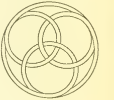



**Ex. 6.** If $ABCD$ is a quadrilateral circumscribed about a circle whose center is $O$, prove that $\angle AOB + \angle COD = 180°$.

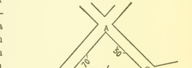
*Suggestion.* — Compare $\angle EOB$ and $\angle BOF$; $\angle EOA$ and $\angle AOH$, etc.




**Ex. 7.** Construct a figure like the one adjoining, using for the equal circles from which it is constructed the radius 1 in. Notice that the circles are tangent circles.



**Ex. 8.** A very small triangular piece of ground $ABC$ lies in the intersection of three streets. Make a drawing to scale ($1'' = 20'$). Then construct corners which will be both more useful and more artistic than the sharp corners. Indicate on your drawing the radii of the circles you construct in the corners.




**Ex. 9.** Prove that an inscribed angle whose intercepted arc is less than a semicircle is an acute angle; and one whose arc is greater than a semicircle is an obtuse angle.

**Ex. 10.** If any number of equal angles are inscribed in an arc, their bisectors pass through a common point.

**Ex. 11.** If the diagonals of an inscribed quadrilateral intersect at the center of the circle, the figure is a rectangle.

**Ex. 12.** Prove that a parallelogram inscribed in a circle is a rectangle.

**Ex. 13.** If $AD$ and $AF$ are tangents to the circle whose center is $O$ and $E$ is any point in major arc $DF$, then, $\angle DEF = 90° - \frac{1}{2} \angle A$.

*Suggestions.* — 1. Draw $OD$ and $OF$.

2. Compare $\angle E$ with $\angle DOF$, and $\angle DOF$ with $\angle A$.

**Ex. 14.** If $AB$ and $AC$ are tangents to a circle whose center is $O$ from a point $A$, touching the circle at $B$ and $C$ respectively, and $D$ is any point on the minor arc $BC$, then $\angle BDC = 90° + \frac{1}{2} \angle A$. (Ex. 71, p. 117.)

**Ex. 15.** $ABCD$ is a quadrilateral inscribed in a circle. Another circle is drawn upon $AD$ as chord, meeting $AB$ and $CD$ at $E$ and $F$ respectively. Prove chords $BC$ and $EF$ parallel. (Ex. 71, p. 117.)

**Ex. 16.** If the opposite angles of a quadrilateral $ABCD$ are supplementary, a circle can be circumscribed about the quadrilateral.

*Suggestions.* — 1. Assume that $D$ falls outside the $\odot$ through $A$, $B$, and $C$, and that the $\odot$ cuts $CD$ at $E$.

2. Derive two contradictory facts about $\angle D$ and $\angle AEC$, using the hypothesis and Ex. 71, p. 117.

3. Next, assume that $D$ falls inside the $\odot ABC$ and complete the indirect proof.

**Ex. 17.** If a right triangle has for its hypotenuse the side of a square and lies outside the square, the straight line drawn from the center of the square to the vertex of the right angle bisects the right angle.

*Suggestion.* — The $\odot$ on the hypotenuse as diameter must pass through the center of the square and also through the vertex of the right angle. (Ex. 16, p. 285.) Draw this circle.

**Ex. 18.** The perpendiculars drawn from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.


*Suggestions.* — 1. Ⓢ can be circumscribed about quadrilaterals $BDOF$, $CDOE$, and $AEOF$. (Ex. 16, p. 285.)

2. Compare $\angle ODF$ with $\angle OBF$, and $\angle ODE$ with $\angle OCE$.

3. Compare $\angle OBF$ with $\angle OCE$, by connecting each with $\angle BAC$.

4. Then $AD$ bisects $\angle$ $EDF$. 5. Similarly for $\angle DEF$ and $\angle DFE$.

**Ex. 19.** Construct the triangle having given the feet of the perpendiculars from the vertices to the opposite sides. (Recall Ex. 18, p. 285.)

**Ex. 20.** If sides $AB$ and $BC$ of inscribed hexagon $ABCDEF$ are parallel to sides $DE$ and $EF$ respectively, prove side $AF$ parallel to side $CD$.

**Ex. 21.** If a circle be drawn upon the radius of another circle as diameter, any chord of the greater circle passing through the point of contact of the circles is bisected by the smaller circle.

*Suggestion.* — Recall § 148.

**Ex. 22.** Prove Prop. XXI, Book II, by drawing through $B$ a chord parallel to $CD$. (Recall § 208.)

**Ex. 23.** If sides $AB$ and $BC$ of inscribed quadrilateral $ABCD$ subtend arcs of 69° and 112° respectively, and $\angle AED$ between the diagonals is 87°, how many degrees are there in each angle of the quadrilateral?

*Suggestions.* — 1. Let $x = \widehat{AD}$ and $y = \widehat{DC}$. Determine these arcs algebraically.

2. Then determine the size of each of the required angles.

**Ex. 24.** Prove Prop. XXII, Book II, by drawing through $B$ a chord parallel to $CD$. (Recall § 208.)

**Ex. 25.** Prove that the measure of the angle between two tangents is the supplement of the measure of the smaller of the two intercepted arcs.

*Suggestion.* — After obtaining the measure of the angle, substitute in it for the larger arc the value of that arc in terms of the smaller arc.

**Ex. 26.** If sides $AB$, $BC$, and $CD$ of an inscribed quadrilateral subtend arcs of 99°, 106°, and 78° respectively, and sides $BA$ and $CD$ extended meet at $E$, and sides $AD$ and $BC$ at $F$, find the number of degrees in $\angle AED$ and $\angle AFB$.

**Ex. 27.** If $\angle$s $A$, $B$, and $C$ of circumscribed quadrilateral $ABCD$ are 128°, 67°, and 112°, respectively, and sides $AB$, $BC$, $CD$, and $DA$ are tangent to the circle at points $E$, $F$, $G$, and $H$ respectively, find the number of degrees in each angle of the quadrilateral $EFGH$.

**Ex. 28.** If $AB$ and $AC$ are the tangents to a circle from a point $A$, and $D$ is any point on the major arc subtended by chord $BC$, prove that $\angle ABD + \angle ACD$ is constant.

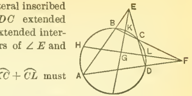
*Suggestions.* — 1. $\angle ABD + \angle ACD = 360° - \angle A - \angle D$. Why?

2. Substitute for $\angle A$ and $\angle D$ their measures.

**Ex. 29.** If $ABCD$ is a circumscribed quadrilateral, prove that the angle between the lines joining the opposite points of contact equals $\frac{1}{2}(\angle A + \angle C)$ or is supplementary to it.

*Suggestion.* — Find the measure of each of the angles. Add the measure of $\angle A$ and $\angle C$.

**Ex. 30.** $ABCD$ is a quadrilateral inscribed in a circle. If sides $AB$ and $DC$ extended intersect at $E$, and $AD$ and $BC$ extended intersect at $F$, prove that the bisectors of $\angle E$ and $\angle F$ are perpendicular.




*Suggestions.* — 1. $\overset{\frown}{AM} + \overset{\frown}{AH} + \overset{\frown}{KC} + \overset{\frown}{CL}$ must $= 180°$.

2. $\overset{\frown}{AM}$ enters in the measure of $\angle AEM$, and $\overset{\frown}{AH}$ in that of $\angle AFH$. Express these measures and add the results. This will give a start on the proof.

Construct the $\triangle\ ABC$ having given:

**Ex. 31.** $a,\ h_b,\ h_c$.

**Ex. 32.** $a,\ h_c,\ t_C$.

**Ex. 33.** $A,\ C,\ t_C$.

**Ex. 34.** $c,\ h_c,\ m_c$.

**Ex. 35.** $a,\ b,\ A$.

**Ex. 36.** $A,\ B,\ h_c$.

**Ex. 37.** $b,\ m_a,\ C$.

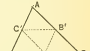
**Ex. 38.** $A,\ t_A,\ h_a$.

**Ex. 39.** Construct a right triangle having given the altitude upon the hypotenuse and one of the legs of the triangle.

**Ex. 40.** Construct a right triangle having given the altitude upon the hypotenuse and one of the acute angles.

**Ex. 41.** Construct a triangle having given the mid-points of its sides.

*Suggestion.* — How does $C'B'$ compare with $BC$?

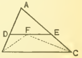



**Ex. 42.** Construct a tangent to an arc of a circle at a given point of the arc without using the center of the circle.

**Ex. 43.** Construct a tangent to a given circle which will be perpendicular to a given straight line.

**Ex. 44.** Given the mid-point of a chord of a circle, construct the chord.

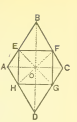
**Ex. 45.** Construct a parallel to side $BC$ of $\triangle\ ABC$ meeting $AB$ and $AC$ at $D$ and $E$ respectively, so that $DE$ will equal the sum of $BD$ and $CE$.




**Ex. 46.** Inscribe a square within a given right triangle having one of its angles coincident with the right angle of the triangle and the opposite vertex lying on the hypotenuse.

*Suggestion.* — In the analysis figure, draw the diagonal from the vertex of the right triangle.

**Ex. 47.** Construct a rhombus within a given triangle, having one angle coincident with an angle of the triangle, and the opposite vertex lying on the opposite side of the triangle.




**Ex. 48.** Construct a square which will have its vertices on the sides of a given rhombus.

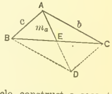
*Suggestion.* — Make an analysis based upon the adjoining figure.

**Ex. 49.** Construct two tangents to a given circle which will make a given angle with the circle.

*Suggestion.* — Draw the given angle at the center of the circle.

**Ex. 50.** Given an angle of a triangle and the segments of the opposite side made by the altitude drawn to that side. Construct the triangle.

**Ex. 51.** Construct a $\triangle ABC$ having given $c$, $b$, and $m_a$. Make an analysis based upon the adjoining figure.

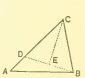



**Ex. 52.** Through a given point outside a circle, construct a secant whose internal and external segments will be equal.

*Suggestion.* — For the analysis figure, connect the center of the circle with the given point, and also with the points of intersection of the secant and the circle. Recall Ex. 51, p. 288.

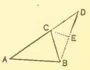
**Ex. 53.** Given the base, an adjacent acute angle, and the difference between the other two sides of the triangle, construct the triangle.

*Suggestions.* — 1. Let $CD = CB$. Then $AD = AC - BC$.
2. Then $\triangle ABD$ can be made the basis of the construction.

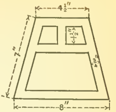



**Ex. 54.** Given the base of a triangle, an adjacent angle, and the sum of the other two sides, construct the triangle.

*Suggestions.* — 1. Let $AD = AC + CB$. Draw $CE \perp BD$.
2. $\triangle ABD$ can be made the basis of the construction.



**Ex. 55.** Construct full size the pattern for the faces of a mission lamp as shown in the adjoining figure, using the dimensions indicated.




**Ex. 56.** Construct a circle tangent to a given line and having its center at a given point not on the line.

**Ex. 57.** Construct a circle which will be tangent to each of two parallels and will pass through a given point lying between the parallels.

**Ex. 58.** Construct a circle having its center in a given line, and passing through two points not in the line.

**Ex. 59.** Construct a circle with given radius which will be tangent to a given circle and will pass through a given point outside of the circle.

**Ex. 60.** Construct a circle with given radius which will be tangent to a given circle and pass through a given point inside of the circle.

**Ex. 61.** Construct a circle with a given radius which will be tangent to a given line and also to a given circle.

**Ex. 62.** Construct a circle which will be tangent to a given circle at a given point on it and also tangent to a given straight line.

**Ex. 63.** Construct a circle which will be tangent to a given circle at a given point on it and also pass through a given point outside of the circle.

## BOOK III

**Ex. 1.** Prove the theorem of § 268 on the hypothesis that $AD : DB = AE : EC$.

*Suggestion.* — Use the same construction. Write the hypothesis by composition, and use § 262.

**Ex. 2.** Let $P$ be any point not in line $AB$ and $R$ any point in $AB$. Let $S$ be a point in segment $PR$, such that $PS : PR = 1 : 3$. Suppose that $R$ moves along $AB$. What is the locus of point $S$?

**Ex. 3.** $XY$ is parallel to the side $AB$ of $\triangle OAB$, meeting $OA$ at $X$ and $OB$ at $Y$. Point $C$ is taken between $X$ and $A$ of $OA$, and $BC$ is drawn. $XZ$ is drawn parallel to $BC$, meeting $YB$ at $Z$. Prove $CY \parallel AZ$.

*Suggestion.* — Try to prove $OC \cdot OZ = OA \cdot OY$; then use § 252.

**Ex. 4.** State and prove the converse of Prop. IV, § 270. (Fig. of Prop. IV. Prove $\angle BAD = \angle CAD$. Extend $CA$ to $E$, making $AE = AB$.)

**Ex. 5.** The sides of a triangle are $a$, $b$, and $c$, respectively. Derive formulæ for the segments of side $c$ made by the bisector of $\angle C$.

**Ex. 6.** $AB$ is the hypotenuse of right $\triangle ABC$. If perpendiculars be drawn to $AB$ at $A$ and $B$, meeting $AC$ extended at $D$, and $BC$ extended at $E$, prove $\triangle s$ $ACE$ and $BCD$ similar.

**Ex. 7.** If altitudes $AD$ and $CE$ of $\triangle ABC$ intersect at $F$, prove $AF : AB = EF : BD$.

**Ex. 8.** $AB$ is a chord of a circle, and $CE$ is any chord drawn through the middle point $C$ of arc $AB$, cutting chord $AB$ at D.

Prove $AC$ is a mean proportional between $CD$ and $CE$.

**Ex. 9.** Two circles are tangent internally at $C$. $CA$ is drawn meeting the smaller circle at $B$ and the larger at $A$; $CE$ is drawn meeting the smaller circle at $D$ and the larger at $E$. Prove $CB : CA = CD : CE$.

**Ex. 10.** The diagonals of a trapezoid, whose bases are $AD$ and $BC$, intersect at $E$. If $AE = 9$, $EC = 3$, and $BD = 16$, find $BE$ and $ED$.

*Suggestion.* — Prove $AE : EC = DE : EB$.

**Ex. 11.** Let $AC$ be the hypotenuse of right $\triangle ABC$, and $E$ and $F$ be any points on $AB$ and $BC$ respectively; let $ED$ and $FG$ be perpendiculars to $AC$, meeting $AC$ at $D$ and $G$ respectively. Prove $AE : FC = ED : GC$. (Recall § 109.)

**Ex. 12.** $\angle A$ of $\triangle ABC$ is a right angle. $DEFG$ is a square having $E$ and $F$ on $BC$, $D$ on $AC$, and $G$ on $AB$. Prove $CE : EF = EF : FB$.

*Suggestion.* — Compare $\triangle CDE$ and $\triangle BFG$.

**Ex. 13.** $AB$ and $AC$ are the tangents to a circle $O$ from point $A$. If $CD$ is drawn perpendicular to $OB$ produced at $D$, then $AB : OB = BD : CD$.

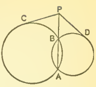
*Suggestion.* — Draw $OA$ and $BC$. Prove $OA \perp BC$.

**Ex. 14.** $\triangle ABC$ is an isosceles triangle. If the perpendicular to $AB$ at $A$ meets base $BC$, extended if necessary, at $E$, and $D$ is the mid-point of $BE$, then $AB$ is the mean proportional between $BC$ and $BD$.

*Suggestion.* — Recall § 284 and Ex. 175, Book I.

**Ex. 15.** Let $r$ be the radius of a circle and $c$ be the distance from the center of the circle to a point $P$ outside the circle. Express the length of the tangent to the circle from $P$, in terms of $r$ and c.

**Ex. 16.** What is the length of the tangent to a circle whose diameter is 16, from a point whose distance from the center is 17?




**Ex. 17.** Prove that the tangents to two intersecting circles from any point in their common chord produced are equal. (Figure adjoining.)

**Ex. 18.** If two circles intersect, their common chord produced bisects their common tangents.

**Ex. 19.** If the altitude be drawn to the hypotenuse of a right triangle, the segments of the hypotenuse have the same ratio as the squares of the adjacent legs.

**Ex. 20.** What is the length of a chord of a circle which is 6 in. from the center, if the radius is 10 in.?

**Ex. 21.** The equal angles of an isosceles triangle are each 30°, and the equal sides are each 8 in. in length. What is the length of the base?

*Suggestion.* — Recall Ex. 128, Book I.

**Ex. 22.** Find the altitude to the base of an isosceles triangle if the base is 8 inches and the sides are each 10 inches in length.

**Ex. 23.** If the equal sides of an isosceles right triangle are each 18 in. in length, what is the length of the median drawn from the vertex of the right angle?

**Ex. 24.** One of the non-parallel sides of a trapezoid is perpendicular to the bases. If the length of this side is 40, and of the parallel sides 31 and 22, respectively, what is the length of the other side?

**Ex. 25.** If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centers?

**Ex. 26.** If $BC$ is the hypotenuse of right triangle $ABC$, prove $(a+b+c)^2 = 2(a+c)(a+b)$.

**Ex. 27.** If the diagonals of a rhombus are $m$ and $n$ respectively, derive a formula for the perimeter of the rhombus.

**Ex. 28.** The diameter which bisects a chord 12 in. long is 20 in. in length. Find the distance from either extremity of the chord to the extremities of the diameter.

*Suggestions.* — 1. Let $x$ represent one segment of the diameter made by the chord. 2. Recall § 289.

**Ex. 29.** The radius of a circle is 16 in. Find the length of the chord which joins the points of contact of two tangents, each 30 in. in length, drawn to the circle from a point outside the circle.

*Suggestions.* — 1. Draw the radii to the points of contact. 2. Recall § 288.

**Ex. 30.** Two parallel chords on opposite sides of the center of a circle are 48 in. and 14 in. long, respectively, and the distance between their mid-points is 31 in. What is the diameter of the circle?

*Suggestion.* — Let $x$ represent the distance from the center to the middle point of one chord, and $31 - x$ the distance from the center to the middle point of the other. Then the square of the radius may be expressed in two ways in terms of $x$.

**Ex. 31.** The parallel sides, $AD$ and $BC$, of a circumscribed isosceles trapezoid are 18 and 6 respectively. Find the diameter of the circle.

*Suggestions.* — 1. Recall Ex. 35, Book II.

2. Through $B$, draw $BE \parallel CD$, meeting $AD$ at $E$.

**Ex. 32.** The diameters of two circles are 12 and 28, respectively, and the distance between their centers is 29. Find the length of the common internal tangent.

*Suggestion.* — Find the $\perp$ drawn from the center of the smaller $\odot$ to the radius of the greater $\odot$ extended through the point of contact.

**Ex. 33.** Prove that the square of the common tangent to two circles which are tangent to each other externally is equal to 4 times the product of their radii.

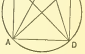
**Ex. 34.** If $D$ is the mid-point of leg $BC$ of right triangle $ABC$, prove that the square of the hypotenuse $AB$ exceeds 3 times the square of $CD$ by the square of $AD$.

**Ex. 35.** If $AB$ is the base of isosceles triangle $ABC$ and $AD$ is perpendicular to $BC$, prove $\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 = 3\,\overline{AD}^2 + 2\,\overline{CD}^2 + \overline{BD}^2$.

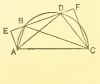
**Ex. 36.** If $D$ is the mid-point of leg $BC$ of right triangle $ABC$, and $DE$ is drawn perpendicular to hypotenuse $AB$, prove $\overline{AE}^2 - \overline{BE}^2 = \overline{AC}^2$.

**Ex. 37.** If in right triangle $ABC$, acute angle $B$ is double acute angle $A$, prove $\overline{AC}^2 = 3\,\overline{BC}^2$.

*Suggestion.* — Recall Ex. 128, Book I.

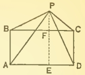



**Ex. 38.** Prove that the sum of the squares of the distances of any point on a circle from the vertices of an inscribed square is equal to twice the square of the diameter of the circle.




**Ex. 39.** If $ABC$ and $ADC$ are angles inscribed in a semicircle, and $AE$ and $CF$ are drawn perpendicular to $BD$ extended, prove

$$\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2.$$




**Ex. 40.** If lines be drawn from any point $P$ to the vertices of rectangle $ABCD$, prove that

$$\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$$

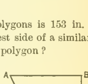
**Ex. 41.** Inscribe in a given circle a triangle similar to a given triangle.

*Suggestion.* — Circumscribe about the given $\triangle$ a $\odot$, and draw radii to the vertices. Recall § 293.

**Ex. 42.** Construct a right triangle having given its perimeter and an acute angle.

*Suggestion.* — Any right triangle containing the given acute angle will be similar to the required triangle. The sides of the required triangle can be determined by § 297.

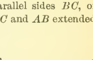
**Ex. 43.** The perimeter of one of two similar polygons is 153 in.; the shortest side of this polygon is 18 in. The shortest side of a similar polygon is 24 in.; what is the perimeter of the second polygon?




**Ex. 44.** The adjoining figure is *similar* to the boundary of an irregular field of a farm; the ratio of similitude of the figure and the boundary of the field is 1 : 2400. Determine the perimeter of the field itself by first finding the perimeter of the adjoining figure and then applying § 297.

**Ex. 45.** If $E$ is the mid-point of one of the parallel sides $BC$, of trapezoid $ABCD$, and $AE$ and $DE$ extended meet $DC$ and $AB$ extended at $F$ and $G$ respectively, then $FG$ is parallel to $BC$.

*Suggestion.* — $GF \parallel BC$ if $GB : GA = FE : FA$.




**Ex. 46.** $\triangle$ $ABC$ and $A'BC$ have their vertices $A$ and $A'$ in a line parallel to their common base $BC$. If a parallel to $BC$ cuts $AB$ at $D$ and $AC$ at $E$, $A'B$ at $D'$ and $A'C$ at $E'$, then $DE = D'E'$.

*Suggestion.* — Prove $DE : BC = D'E' : BC$.

**Ex. 47.** If $AB$ and $CD$ are equal and parallel segments, prove that $p_m^{AB}$ equals $p_m^{CD}$, where $m$ is any line.

**Ex. 48.** If $AD$ and $BE$ are the perpendiculars from vertices $A$ and $B$, respectively, of acute-angled triangle $ABC$ to the opposite sides, prove

$$AC \times AE + BC \times BD = \overline{AB}^2.$$

*Suggestion.* — Find $2\,AC \times AE$ by § 310, and in like manner find $2\,BC \times BD$. Then add.

**Ex. 49.** In triangle $ABC$, if angle $C$ equals 120°, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + BC \times AC.$$

*Suggestion.* — Recall § 311.

**Ex. 50.** If a line be drawn from vertex $C$ of isosceles triangle $ABC$, meeting base $AB$ extended at $D$, prove $\overline{CD}^2 - \overline{CB}^2 = AD \times BD$.

*Suggestion.* — Apply § 311 in $\triangle BCD$.

**Ex. 51.** From the conclusion of § 311, derive a formula for $p_a^2$ in terms of $a$, $b$, and $c$.

**Ex. 52.** In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.


**Prove** $AB \times AC = DB \times DC - AD^2$.

*Suggestions.* — 1. The solution is similar to that of § 318.
2. First prove $\triangle ABD \sim \triangle ACE$.

**Ex. 53.** $DEFG$ is a square having its vertices $D$ and $E$ on sides $AB$ and $BC$ respectively of triangle $ABC$ and its vertices $F$ and $G$ on side $AC$. Let $BH$ be $\parallel$ to $AC$, meeting $AE$ extended at $G$; let $HK$ be $\perp AC$ and $BT \perp AC$. Prove $BHKT$ is a square.

**Ex. 54.** In a given triangle, construct a square which shall have two vertices lying on one side of the triangle and having its other two vertices on the other two sides of the triangle, one on each side.

**Ex. 55.** Construct a square which will have two of its vertices on a diameter of a given circle, and the remaining two vertices on the semicircle constructed on this diameter.

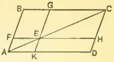
**Ex. 56.** Circumscribe about a given circle a triangle similar to a given triangle.

*Suggestion.* — Inscribe in the given triangle a circle and draw radii to the points of tangency.

## BOOK IV

**Ex. 1.** The sides of a triangular field are 10 rd., 8 rd., and 9 rd. respectively. Make a scale drawing of the boundary of the field on coördinate paper, and estimate the area of the field.

**Ex. 2.** Angle $B$ of $\triangle ABC$ is a right $\angle$. $D$ and $E$ are the mid-points of $AB$ and $AC$ respectively. $CF$, perpendicular to $BC$ at $C$, meets $DE$ extended at $F$. Prove $\triangle ABC = \square BCFD$.




**Ex. 3.** $E$ is any point on diagonal $AC$ of ▱ $ABCD$. Through $E$, parallels to $AD$ and $AB$ are drawn, meeting $AB$ and $CD$ at $F$ and $H$ respectively, and $BC$ and $AD$ at $G$ and $K$ respectively.

Prove ▱ $FBGE$ = ▱ $EHDK$.

**Ex. 4.** All the lots of a certain city block are rectangular and 125 ft. in depth (from front to back). Compare two lots $A$ and $B$ if the frontage of Lot $A$ is 40 ft. and that of Lot $B$ is 60 ft. (Do not obtain their areas.)

**Ex. 5.** Two rectangles $R_1$ and $R_2$ have equal altitudes.

(*a*) What part of $R_2$ is $R_1$ if the base of $R_1$ is 5 and the base of $R_2$ is 8?

(*b*) What is the ratio of $R_1$ to $R_2$ if the bases are 25 and 10 respectively?

**Ex. 6.** Divide a given triangle into three equal parts by lines drawn through one of its vertices.

**Ex. 7.** Determine the area of the triangle whose sides are 25, 17, and 28.

**Ex. 8.** If $b$ is the base and $s$ is one of the equal sides of an isosceles triangle, prove that the area is $\frac{1}{4}b\sqrt{4s^2 - b^2}$.

**Ex. 9.** The area of an isosceles right triangle is 81 sq. in. Determine its hypotenuse.

*Suggestion.* — Let $x$ represent one of the sides. Determine $x$ and then determine the hypotenuse.

**Ex. 10.** The area of an equilateral triangle is $9\sqrt{3}$. Determine its side.

*Suggestion.* — Use the formula proved in Ex. 29, Book IV.

**Ex. 11.** The altitude of an equilateral triangle is 3. Determine its area.

*Suggestion.* — Let $x$ represent one side. Determine $x$ and then determine the area.

**Ex. 12.** The area of an equilateral triangle is $16\sqrt{3}$. Determine its altitude.

**Ex. 13.** The area of a rhombus is 240 sq. in. and its side is 17 in. Find its diagonals.

*Suggestion.* — Represent the diagonals by $2x$ and $2y$. Proceed algebraically.

**Ex. 14.** One diagonal of a rhombus is five thirds the other; the difference of the diagonals is 8 in. Determine the area of the rhombus.

**Ex. 15.** The segments of the hypotenuse of a right triangle made by the altitude drawn to the hypotenuse are $5\frac{2}{5}$ and $9\frac{3}{5}$ respectively. Determine the area of the triangle. (§ 288.)

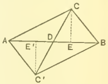
**Ex. 16.** The sides of $\triangle ABC$ are $AB = 13$, $BC = 14$, and $AC = 15$. Bisector $AD$ of $\angle A$ meets $BC$ at $D$. Find the areas of $\triangle\triangle ABD$ and $ACD$.

*Suggestion.* — 1. Compute the altitude $h_a$. (§ 313.)
2. Determine $BD$ and $DC$ by § 270.

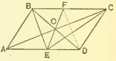
**Ex. 17.** If $D$ and $E$ are the mid-points of sides $BC$ and $AC$ respectively of $\triangle ABC$, prove $\triangle ABD = \triangle ABE$.

*Suggestion.* — Compare the altitudes to $AB$ from $D$ and $E$.

**Ex. 18.** If diagonal $AC$ of quadrilateral $ABCD$ bisects diagonal $BD$, then $\triangle ABC = \triangle ADC$.

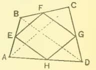



**Ex. 19.** Two equal triangles have a common base, and lie on opposite sides of it. Prove that the base, extended if necessary, bisects the line joining their vertices. (Prove $CD = C'D$.)

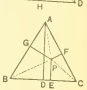



**Ex. 20.** If $EF$ is any straight line drawn through the point of intersection of the diagonals of $\square ABCD$, meeting sides $AD$ and $BC$ at $E$ and $F$ respectively, then $\triangle BEF = \triangle CED$.

*Suggestion.* — Does $BF = ED$ ?




**Ex. 21.** If $E$, $F$, $G$, and $H$ are the mid-points of sides $AB$, $BC$, $CD$, and $DA$, respectively, of quadrilateral $ABCD$, prove $EFGH$ a parallelogram equal to one half $ABCD$.

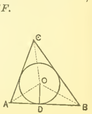



**Ex. 22.** Prove that the sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

*Suggestions.* — 1. $\triangle BPC + \triangle BPA + \triangle APC = \triangle ABC$.
2. Express the area of each triangle and substitute in this equation.

**Ex. 23.** If $E$ is any point in side $BC$ of $\square ABCD$, and $DE$ is drawn, meeting $AB$ extended at $F$, prove $\triangle ABE$ equals $\triangle CEF$.

*Suggestion.* — Compare $\triangle FCD$ with $\square ABCD$.




**Ex. 24.** Prove that the area of a triangle is equal to one half the product of its perimeter by the radius of the inscribed circle.

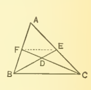
**Ex. 25.** A circle whose diameter is 12 is inscribed in a quadrilateral whose perimeter is 50. Find the area of the quadrilateral.

**Ex. 26.** If the sides of a triangle are 15, 41, and 52, determine the radius of the inscribed circle.

*Suggestions.* — 1. Find the area of the triangle.
2. Make use of the fact proved in Ex. 24.

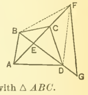
**Ex. 27.** If $D$ is the mid-point of side $BC$ of $\triangle ABC$, $E$ the mid-point of $AD$, $F$ of $BE$, and $G$ of $CF$, then $\triangle ABC = 8 \triangle EFG$.

*Suggestion.* — Draw $EC$.




**Ex. 28.** If $BE$ and $CF$ are medians drawn from vertices $B$ and $C$ of $\triangle ABC$, intersecting at $D$, prove $\triangle BCD$ equals quadrilateral $AEDF$.

*Suggestion.* — Compare $\triangle ABE$ with $\triangle BEC$ and with $\triangle BFC$.




**Ex. 29.** Any quadrilateral $ABCD$ is equivalent to a triangle, two of whose sides are equal to diagonals $AC$ and $BD$, respectively, and include an angle equal to either of the angles between $AC$ and $BD$.

Prove $\triangle EFG = ABCD$, where $EF = AC$, and $EG = BD$.

*Suggestion.* — Compare $\triangle DFG$ with $\triangle BEF$ and then with $\triangle ABC$.

**Ex. 30.** Prove that two triangles are equal if two sides of one equal respectively two sides of the other and the included angles are supplementary.

*Suggestion.* — Place the triangles so that the supplementary angles are adjacent and so that one pair of equal sides coincide.

**Ex. 31.** On coördinate paper, draw the pentagon whose vertices are

$$A = 0, 0;\ B = 5, 0;\ C = 8, 3;\ D = 4, 9;\ E = 0, 6.$$

Determine its approximate area as in Ex. 1 and Ex. 2, Book IV. Construct a $\triangle$ equal to the pentagon. Then construct its base and altitude, and compute its approximate area.

**Ex. 32.** If, in the figure of Prop. X, $AB = 9$ in., $A'B' = 7$ in., and the area of $\triangle A'B'C'$ is 147 sq. in., find the area of $\triangle ABC$.

**Ex. 33.** The area of a certain triangle is $\frac{4}{9}$ the area of a similar triangle. If the altitude of the first is 4 ft., what is the altitude of the second?

**Ex. 34.** If, in § 343, area of $\triangle A'B'C' = 147$ sq. in., $AB = 9$ in., and $A'B' = 3$ in., find the area of $\triangle ABC$.

**Ex. 35.** The sides $AB$ and $AC$ of $\triangle ABC$ are 15 and 22, respectively. From a point $D$ in $AB$, a parallel to $BC$ is drawn meeting $AC$ at $E$, and dividing the triangle into two equal parts. Find $AD$ and $AE$.

**Ex. 36.** If similar polygons be drawn upon the legs of a right triangle as homologous sides, the polygon drawn upon the hypotenuse is equal to the sum of the polygons drawn upon the legs.

*Suggestions.* — 1. Compare the polygon on each leg with the one on the hypotenuse by § 344.

2. Add the resulting equations and simplify.

**Ex. 37.** Construct a triangle similar to two given similar triangles and equal to their sum.

**Ex. 38.** Two similar triangles have homologous sides of 8 in. and 15 in. respectively. Find the homologous side of a similar triangle equal to their sum.

**Ex. 39.** Construct a triangle similar to two similar triangles, and equal to their difference.

**Ex. 40.** If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon whose homologous side is 10 in.?

**Ex. 41.** If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the homologous side of a similar polygon whose area is 392 sq. ft.?

**Ex. 42.** Construct a rectangle having a given altitude and equal to a given parallelogram.

*Suggestion.* — Recall Ex. 78, Book IV.

**Ex. 43.** Construct a parallelogram equal to a given parallelogram and having two adjacent sides equal to given segments $m$ and $n$ respectively.

**Ex. 44.** Construct a parallelogram equal to a given parallelogram and having one side equal to a given segment $m$, and one diagonal equal to a given segment $n$.

**Ex. 45.** Construct a right triangle equal to a given square, having given its hypotenuse.

*Suggestion.* — Determine the altitude to the hypotenuse as in Ex. 78, Book IV; then construct the triangle, using the methods of § 241.

**Ex. 46.** Construct a right triangle equal to a given triangle, having given its hypotenuse.

## BOOK V

**Ex. 1.** Prove that the diagonals drawn from one vertex of a regular polygon having $n$ vertices to each of the other vertices divides the angle at that vertex into $(n-2)$ equal parts.

**Ex. 2.** Prove that the central angle of any regular polygon is the supplement of the vertex angle of the polygon.

**Ex. 3.** Prove that the sum of the perpendiculars drawn from any point within a regular polygon to the sides of the polygon is equal to the apothem multiplied by the number of sides of the polygon.

*Suggestions.* — Connect the point with each vertex. Notice that the sum of the triangles so formed equals the polygon. Express the area of each triangle and form an equation.

**Ex. 4.** In the figure for § 365 prove that:

(*a*) $s_4 > s_8 > s_{16}$, etc. (See § 362.)

(*b*) $a_4 < a_8 < a_{16}$, etc.

(*c*) $k_4 < k_8 < k_{16}$, etc.

**Ex. 5.** Prove that an equiangular polygon inscribed in a circle is regular if the number of sides is odd.

**Ex. 6.** Prove that an equiangular polygon circumscribed about a circle is regular.

*Suggestions.* — 1. Draw the chords joining the points of tangency.

2. Prove the resulting $\triangle$s:

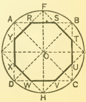
(*a*) are isosceles; (*b*) are mutually equiangular; (*c*) that $XY = YZ$, etc. See diagram in § 367.

Complete the proof.

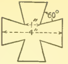
**Ex. 7.** Repeat Ex. 14, p. 227, for a regular octagon circumscribed about a circle of radius 10.

**Ex. 8.** Prove that diagonal $AE$ of regular octagon $ABCDEFGH$ is the perpendicular-bisector of diagonal $BH$.

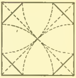
**Ex. 9.** In the adjoining figure, $ABCD$ and $EFGH$ are squares inscribed in the circle, such that $AF = FB = BG$, etc. Is $RSTUVWXY$ a regular octagon?



**Ex. 10.** Construct a Maltese cross having the dimensions indicated.




**Ex. 11.** Prove that the construction indicated in the adjoining figure serves to inscribe a regular octagon in the square.



**Ex. 12.** A regular octagon is inscribed in a circle of radius 10. Compute $s_8$, $p_8$, $a_8$, and $k_8$.

**Ex. 13.** Prove that for a regular octagon inscribed in a circle of radius $R$:

(a) $s_8 = R\sqrt{2-\sqrt{2}}$; (c) $a_8 = \frac{R}{2}\sqrt{2+\sqrt{2}}$;

(b) $p_8 = 8R\sqrt{2-\sqrt{2}}$; (d) $k_8 = 2R^2\sqrt{2}$.

**Ex. 14.** Construct a regular octagon having its sides 1 inch long.

**Ex. 15.** What is the relation between the area of the inscribed and of the circumscribed equilateral triangles of a given circle?

**Ex. 16.** What is the relation between the perimeter of the inscribed and of the circumscribed equilateral triangles of a given circle?

**Ex. 17.** A regular hexagon is inscribed in a circle of radius $r$. Prove:

(a) $s_6 = r$; (b) $a_6 = \frac{r\sqrt{3}}{2}$; (c) $p_6 = 6r$; (d) $k_6 = \frac{3r^2\sqrt{3}}{2}$.

**Ex. 18.** A regular triangle is inscribed in a circle of radius $r$. Prove:

(a) $s_3 = r\sqrt{3}$; (b) $a_3 = \frac{1}{2}r$; (c) $p_3 = 3r\sqrt{3}$; (d) $k_3 = \frac{3r^2\sqrt{3}}{4}$.

**Ex. 19.** Prove that the apothem of an equilateral triangle is one third the altitude of the triangle.

**Ex. 20.** (a) In a circle of radius 2.5 in., inscribe a regular hexagon.

(b) Also inscribe in the same circle a regular triangle and a regular 12-gon.

(c) Prove that $s_3 > s_6 > s_{12}$, etc.

(d) Prove that $a_3 < a_6 < a_{12}$, etc.

(e) Prove that $p_3 < p_6 < p_{12}$, etc.

(f) Prove that $k_3 < k_6 < k_{12}$, etc.

**Ex. 21.** What is the perimeter and area of a regular circumscribed hexagon about a circle of radius 10?

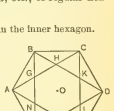
**Ex. 22.** Repeat the foregoing exercise for a circle of radius $R$.

**Ex. 23.** What is the perimeter and the area of a regular triangle circumscribed about a circle of radius 10?

**Ex. 24.** Repeat the foregoing exercise for a circle of radius $R$.

**Ex. 25.** Prove that the diagonals $AC$, $BD$, $CE$, etc., of regular hexagon $ABCDEF$ form another regular hexagon.

*Suggestion.* — Prove that a circle can be inscribed in the inner hexagon.




**Ex. 26.** Prove that the area of the inner hexagon of the foregoing exercise is one third the area of $ABCDEF$.

*Suggestion.* — Express the area of each polygon in terms of the radius $OB$ of $ABCDEF$.

**Ex. 27.** Prove that the area of a regular inscribed hexagon is a mean proportional between the areas of an inscribed and of a circumscribed equilateral triangle.

*Suggestion.* — Express the areas of each in terms of the radius.

**Ex. 28.** In a given equilateral triangle, inscribe a regular hexagon having two of its vertices lying on each side of the triangle.

**Ex. 29.** Construct a regular hexagon having given one of the diagonals joining two alternate vertices.

**Ex. 30.** A square is inscribed in an equilateral triangle whose side is $a$, having two vertices in one side of the triangle, and one in each of the other sides. Compute the area of the square.

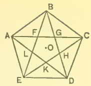
**Ex. 31.** A regular 12-gon is inscribed in a circle of radius $R$. Prove:

(a) $s_{12} = R\sqrt{2-\sqrt{3}}$;
(c) $p_{12} = 12\,R\sqrt{2-\sqrt{3}}$;
(b) $a_{12} = \frac{R}{2}\sqrt{2+\sqrt{3}}$;
(d) $k_{12} = 3\,R^2$.

**Ex. 32.** If the diagonals $AC$ and $BE$ of regular pentagon $ABCDE$ intersect at $F$, prove that $BE = AE + EF$.

**Ex. 33.** Prove that the figure $FGHKL$ formed by parts of the diagonals of regular inscribed petagon $ABCDE$ is also a regular pentagon.
(See figure on page 302.)

*Suggestions.* — Prove that a circle can be inscribed in $FGHKL$.




**Ex. 34.** — In the figure of Prop. VIII, Book V, prove that $OM$ is the side of a regular pentagon inscribed in the circle which can be circumscribed about $\triangle OBM$.

*Suggestion.* — How large is $\angle OBM$?

**Ex. 35.** Construct a regular pentagon having given one of its sides.

**Ex. 36.** Construct a regular pentagon having given one of its diagonals.

**Ex. 37.** If $R$ represents the radius of the circle circumscribed about a regular decagon, prove:

(*a*) $s_{10} = \frac{R}{2}(\sqrt{5} - 1)$;  (*c*) $p_{10} = 5R(\sqrt{5} - 1)$;

(*b*) $a_{10} = \frac{R}{4}\sqrt{10 + 2\sqrt{5}}$;  (*d*) $k_{10} = \frac{5R^2}{4}\sqrt{10 - 2\sqrt{5}}$.

**Ex. 38.** Find the area of the circle inscribed in a square whose area is 25.

**Ex. 39.** If the radius of a circle is $3\sqrt{3}$, what is the area of the sector whose central angle is 150°?

**Ex. 40.** Find the radius of the circle equal to a square whose side is 10.

**Ex. 41.** Find the radius of the circle whose area is one half the area of the circle whose radius is 15.

**Ex. 42.** Find the area of the square inscribed in the circle whose area is $196\pi$ sq. in.

**Ex. 43.** The area of one circle is $\frac{25}{9}$ the area of another. Find the radius of the second if the area of the first is 15.

**Ex. 44.** The side of a square is 8. Find the circumference of its inscribed and circumscribed circles.

**Ex. 45.** The side of an equilateral triangle is 6. Find the area of its inscribed and circumscribed circles.

**Ex. 46.** The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

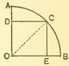
**Ex. 47.** If the apothem of a regular hexagon is 6, what is the area of its circumscribed circle?

**Ex. 48.** Two plots of ground, one a square and one a circle, each contain 70686 sq. ft. How much greater is the perimeter of the square than the length of the circle?

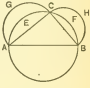
**Ex. 49.** The perimeter of a regular hexagon circumscribed about a circle is $12\sqrt{3}$. What is the circumference of the circle?

**Ex. 50.** The length of the arc subtended by the side of a regular inscribed 12-gon is $\frac{1}{3}\pi$ in. What is the area of the circle?

**Ex. 51.** If the length of a quadrant is 1, what is the diameter of the circle?




**Ex. 52.** Prove that the area of the square inscribed in a sector whose central angle is a right angle, is equal to one half the square on the radius.




**Ex. 53.** If a circle is circumscribed about a right triangle, and on each of the legs of the triangle as diameters semicircles are drawn, exterior to the triangle, the sum of the areas of the crescents thus formed equals the area of the triangle.

Prove area $AECG$ + area $BFCH$ = area $\triangle ABC$.

*Suggestion.* — From the sum of $\triangle ABC$ and the semicircles on $AC$ and $BC$, subtract the semicircle on $AB$. Express each area in terms of sides $a$, $b$, and $c$ of the triangle.

**Ex. 54.** Construct three equal circles having the vertices of an equilateral triangle as their centers and for their radii one half the side of the triangle. Compute the area of that part of the interior of the triangle which is exterior to each of the circles, if the length of the side of the triangle is $s$.

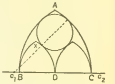
**Ex. 55.** Upon a segment $AC$ draw a semicircle. Upon $AC$ locate a point $B$, not the center of $AC$. Upon $AB$ and $BC$ as diameters draw semicircles within the one drawn upon $AC$ as diameter. Prove that the area of the surface lying within the largest semicircle and exterior to the smaller ones equals the area of the circle drawn upon $BD$ as diameter, where $BD$ is the perpendicular to $AC$ at $B$ meeting the largest semicircle at $D$. (Due to Archimedes.)

**Ex. 56.** With the vertices of an equilateral triangle as centers and the side of the triangle as radius, three equal circles are drawn. Determine the area of that figure which is common to the three circles.

**Ex. 57.** Express in terms of the radius $R$ the area of the segment of a circle whose chord is a side of the inscribed square.

**Ex. 58.** Repeat Ex. 57 if the chord is the side of the equilateral inscribed triangle.

**Ex. 59.** The arch $ABC$ is a *lancet arch*. It consists of two arcs with equal radii, drawn from centers $C_1$ and $C_2$ outside the span $BC$. Within the arch are two other lancet arches.




Let $BC = 2a$; let $C_1C_2 = s$; let $BD = DC$.

(*a*) Determine the height $h$ of the arch $BAC$.

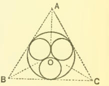
(*b*) What is the length of the radius of the arc $XD$?

(*c*) What is the height of the arch $BXD$?

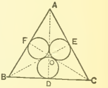
(*d*) What is the radius of the circle indicated as tangent to the arches?

(*e*) What is the area and circumference of the circle?

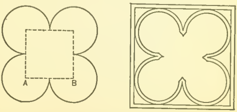
**Ex. 60.** In a given circle, inscribe three equal circles, tangent to each other and to the given circle.




**Ex. 61.** In a given equilateral triangle, inscribe three equal circles, tangent to each other and each tangent to one and only one side of the triangle.




**Ex. 62.** The figure below at the left is a *quatrefoil*.




(*a*) Construct such a figure based upon a square whose side is 2 in.

(*b*) What is the length of the curved line if $AB = s$ inches?

(*c*) What is the area within the curved line if $AB = s$ inches?

(*d*) Notice that the quatrefoil is used in the adjoining design.

**Ex. 63.** Construct a figure like Fig. 1, below, upon a square of side 2 in.

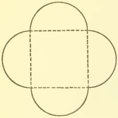


Fig. 1

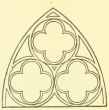


Fig. 2

(*a*) What is the length of the curved line when the side of the square is $s$ inches?

(*b*) What is the total area within the curved line when the side of the square is $s$ inches?

(*c*) Notice that the curved line of Fig. 1 is the fundamental unit of the adjoining window design.

# SOLID GEOMETRY

## BOOK VI

### LINES AND PLANES — POLYEDRAL ANGLES

**442. Surfaces.** No satisfactory elementary definition of surface in general can be given. The surface of a physical object is that part of the object which can, in general, be touched; it separates the portion of space occupied by the object from surrounding space.

The surface of a small pond on a calm day is approximately a *plane surface*.

The surface of a croquet ball or of a billiard ball is a *spherical surface*.

The surface of a "round" marble column is a *cylindrical surface*.

**Ex. 1.** If two points on the surface of a ball were joined by a straight line, where would the line lie?

**Ex. 2.** Are there any two points on the surface of a ball such that the straight line through them lies upon the surface of the ball?

**Ex. 3.** Are there two points on the surface of a cylindrical column such that the straight line joining them lies on the surface of the column?

**Ex. 4.** Does the straight liue joining *every* pair of points on the surface of a cylindrical column lie upon the surface of the column?

**443.** A **Plane** is a surface such that the straight line joining any two points of it lies wholly in it.

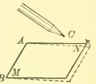
A plane is represented to the eye by a quadrilateral like the figure adjoining.


The plane may be referred to as plane $ABCD$, as plane $AC$, or as plane $M$.

Pupils will find it convenient at times to represent a plane by a thin card.

**Ex. 5.** In plane geometry, it was agreed that a straight line is indefinite in extent. Do you think we should agree that a plane is indefinite in extent? Why?

**Ex. 6.** Let $MN$ represent a thin card of which $AB$ is an edge. Let $C$ represent the "point" of a pencil lying above $MN$. Let $MN$ be turned about $AB$ as an axis in the direction indicated by the arrow.




Will the card eventually come in contact with point $C$?

If $AB$ and $C$ are kept stationary, will the card come in contact with $C$ in more than one position of the card?

**444.** A **plane** is **determined** by a combination of points and lines if it is the only plane which contains those points and lines.

Points and lines lying in the same plane are said to be **Co-planar.**

**445. Postulate.** *A plane can be extended indefinitely.*

**446. Axiom.** *A plane is determined by three non-collinear* points.*

**Ex. 7.** Do two straight lines drawn at random necessarily lie in a plane? Illustrate by holding two pencils.

**Ex. 8.** Do four points usually lie in a plane? Select four in the schoolroom that do and four that do not.

**Ex. 9.** Are two straight lines in space which do not meet no matter how far they are extended parallel?

**Ex. 10.** Why is a tripod used as mounting for a camera or a surveyor's instrument?

**Ex. 11.** Why does a stool with three legs stand firmly whereas one with four legs cannot always be made to stand firmly?

**Ex. 12.** Prove that a plane and a straight line not lying in the plane can have only one common point.

* Non-collinear points are points which do not all lie in one straight line.

## Proposition I. Theorem

**447.** *A plane is determined by*

I. *A straight line and a point outside the line.*

II. *Two intersecting straight lines.*

III. *Two parallel straight lines.*

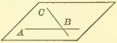
I. **Hypothesis.** Point $C$ lies outside st. line $AB$.

**Conclusion.** $C$ and $AB$ determine a plane.

**Proof.** 1. Points $A$, $B$, and $C$ lie in one and only one plane. § 446

2. $AB$ lies in that plane. § 443

3. $\therefore AB$ and $C$ determine a plane. § 444



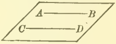
II. **Hypothesis.** $AB$ and $BC$ are intersecting st. lines.

**Conclusion.** $AB$ and $BC$ determine a plane.

**Proof.** 1. $AB$ and point $C$ determine a plane. § 447, I

2. $BC$ lies in that plane. § 443

3. $\therefore AB$ and $BC$ determine a plane.



III. **Hypothesis.** $AB$ and $CD$ are parallel straight lines.

**Conclusion.** $AB$ and $CD$ determine a plane.

**Proof.** 1. $AB$ and $CD$ lie in a plane. § 89

2. $AB$ and $CD$ cannot lie in more than one plane, for, if they did, points $A$, $B$, and $C$ would lie in more than one plane, which is impossible. Why?

3. $\therefore AB$ and $CD$ determine a plane.

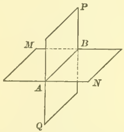
**448.** The **Intersection** of two surfaces or of a surface and a line consists of all points common to the surfaces, or to the surface and the line.

**449.** If a straight line intersects a plane, the point of intersection of the line and the plane is called the **Foot** of the line.

**450. Axiom.** *If two planes intersect, they have at least two common points.*

## Proposition II. Theorem

**451.** *The intersection of two planes is a straight line.*




**Hypothesis.** $A$ and $B$ are two points common to planes $MN$ and $PQ$.

**Conclusion.** The intersection of $MN$ and $PQ$ is a straight line.

**Proof.** 1. Draw straight line $AB$.

2. $AB$ lies in plane $MN$ and also in plane $PQ$. Why?

3. No point outside $AB$ can be in both $MN$ and $PQ$, for, if there were, $MN$ and $PQ$ would coincide. § 447, I

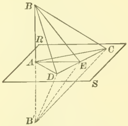
4. Hence the complete intersection of $MN$ and $PQ$ is the straight line $AB$. § 448

**452.** A **line** is **perpendicular to a plane** if it is perpendicular to every line in the plane passing through its foot.

The **plane** is also **perpendicular to the line.**

## Proposition III. Theorem

**453.** *If a line is perpendicular to each of two intersecting lines at their intersection, it is perpendicular to their plane.*




**Hypothesis.** $AD$ and $AC$ intersect at $A$, determining plane $RS$.

$BA \perp AD$ and $BA \perp AC$.

**Conclusion.** $BA \perp$ plane $RS$.

**Proof.** 1. Let $AE$ be any other straight line in $RS$ through $A$. Let $DC$ be a straight line in $RS$ intersecting $AD$, $AE$, and $AC$, at $D$, $E$, and $C$, respectively.

2. Extend $BA$ to $B'$, making $B'A = BA$. Draw $BD$, $BE$, $BC$, $B'D$, $B'E$, and $B'C$.

3. $AD$ and $AC$ are $\perp$ bisectors of $BB'$. Why?

4. $\therefore BD = B'D$ and $BC = B'C$. Prove it.

5. $\therefore \triangle BDC \cong \triangle B'DC$. Prove it.

6. Revolve $\triangle B'DC$ on $DC$ as axis until $B'$ falls on $B$.

7. Then $B'E \equiv BE$. Why?

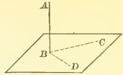
8. $\therefore A$ and $E$ are both equidistant from $B$ and $B'$.

9. $\therefore AE \perp BB'$, or $BB' \perp AE$. § 77

10. But $AE$ is *any* st. line in $RS$ through $A$.

11. $\therefore AB \perp$ every st. line in $RS$ through $A$, and hence $AB \perp$ plane $RS$. § 452

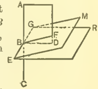
**Ex. 13.** If a line is perpendicular to a line of a plane, is it perpendicular to the plane?




**454. Cor. 1.** *Through a point of a line a plane can be drawn perpendicular to the line.*

*Suggestion.* — Draw $BC$ and $BD$, any two perpendiculars to $AB$.

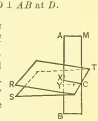
**Note.** — *Through a point of a line only one plane can be drawn perpendicular to the line.* If $ME$ and $RE$ were both $\perp$ to $AB$ at $B$, a plane $AD$ through $AB$ would intersect $ME$ and $RE$ in two lines $BF$ and $BD$, each $\perp$ to $AB$ at $B$. But this is impossible, for, in a plane ($AD$), only one line can be drawn perpendicular to a given line at a point of the line.



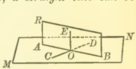

**455. Cor. 2.** *Through a point outside a line, a plane can be drawn perpendicular to the line.* (See Fig. § 454.)

*Suggestion.* — Draw $CB \perp AB$ from $C$; then draw $BD \perp AB$ at $D$.

**Note.** — *Through a point outside a line, only one plane can be drawn perpendicular to the line.* If planes $RT$ and $ST$ through $C$ were both $\perp$ to $AB$, the plane $ABC$ determined by $AB$ and $C$, would intersect $RT$ and $ST$ in lines $XC$ and $YC$, each $\perp$ to $AB$. But this is impossible, for, *in a plane*, only one line can be drawn perpendicular to a given line from a point outside the line.




**456. Cor. 3.** *At a point in a plane, a straight line can be drawn perpendicular to the plane.*



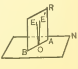

**Construction.** 1. Draw $CD$ any line in $MN$ through $O$.

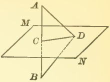
2. Draw plane $RB \perp$ to $CD$ at $O$, meeting $MN$ in line $AB$.

3. Draw $EO$ in plane $RB$, $\perp$ to $AB$ at $O$.

**Statement.** $EO \perp$ plane $MN$ at $O$.

**Note.** — *At a point of a plane, only one straight line can be drawn perpendicular to the plane.* If $EO$ and $E'O$ were both $\perp$ to plane $MN$ at $O$, they would determine a plane $RB$ which would intersect $MN$ in line $AB$. $EO$ and $E'O$ would both be $\perp$ to $AB$ at $O$, and that is impossible. Why?



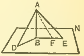


**457. Cor. 4.** *Any point in the plane which is perpendicular to a segment at its mid-point is equidistant from the ends of the segment.*

**Ex. 14.** Each of three concurrent lines is perpendicular to each of the other two. Prove that each is perpendicular to the plane of the other two.

**Ex. 15.** If two oblique lines, drawn to a plane from a point in a perpendicular to the plane, cut off equal distances from the foot of the perpendicular, they are equal.

**Ex. 16.** State and prove the converse of Ex. 15.



**Ex. 17.** If two oblique lines, drawn to a plane from a point in a perpendicular to the plane, cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

If $BE > BD$, prove $AE > AD$.

*Suggestion.*—Take $BF = BD$, and draw $AF$. Recall § 165

**Ex. 18.** State and prove the converse of Ex. 17.

*Suggestion.*—Give an indirect proof, basing it upon Ex. 15 and 17.

**Ex. 19.** If a circle be drawn in a plane and at its center a perpendicular to the plane be erected, any point in this perpendicular is equidistant from the points of the circle.

**Ex. 20.** A line segment of fixed length, having one extremity at a fixed point lying outside a plane, has its other extremity in the plane. What is the locus of the extremity which lies in the plane?

**Ex. 21.** How many different planes can be passed through one straight line?

**Ex. 22.** How many different planes are determined by:

(*a*) Three concurrent lines which do not all lie in one plane?

(*b*) Three parallel lines which do not all lie in one plane?

(*c*) Two intersecting lines and a point which does not lie in their plane?

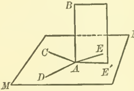
(*d*) Four points, no three of which are collinear and which do not all lie in one plane?

**Ex. 23.** Prove that two parallels and any transversal of them are co-planar.

**Ex. 24.** How many lines of intersection are determined, in general, by three planes?

## Proposition IV. Theorem

**458.** *All the perpendiculars to a straight line at a point of the line lie in a plane perpendicular to the line at the point.*




**Hypothesis.** $AC$, $AD$, and $AE$ are any three ⊥s to $AB$ at $A$.

**Conclusion.** $AC$, $AD$, and $AE$ lie in a plane ⊥ to $AB$ at $A$.

**Proof.** 1. Let $AC$ and $AD$ determine plane $MN$.

2. $\therefore AB \perp$ plane $MN$. Why?

3. Let $AB$ and $AE$ determine plane $ABE$, intersecting $MN$ in $AE'$.

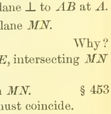
4. $\therefore AB \perp AE'$, since $AE'$ is in $MN$. § 453

5. $\therefore AE$ and $AE'$, both in plane $BE'$, must coincide.

[*In a plane*, only one ⊥ can be drawn to a line at a point in the line.] § 81

6. $\therefore AE$ must lie in $MN$. Why?

7. Hence all ⊥s to $AB$ at $A$ must lie in $MN$. Why?



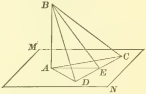

**459. Cor. 1.** *Any point equidistant from the ends of a segment lies in the plane perpendicular to the segment at its midpoint.*

**460. Cor. 2.** *The locus of points in space equidistant from the ends of a segment is the plane perpendicular to the segment at its mid-point.*

*Suggestion.* — Review, if necessary, § 229 of the Plane Geometry and apply §§ 457 and 459.

## Proposition V. Theorem

**461.** *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*

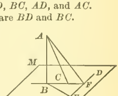



**Hypothesis.** $AB \perp$ plane $MN$; $CD$ is any line in $MN$; $AE \perp CD$; $BE$ is drawn from any point $B$ of $AB$ to $E$.

**Conclusion.** $BE \perp CD$.

*Suggestions.*—1. Take $CE = DE$, and draw $BD$, $BC$, $AD$, and $AC$.
2. Compare $AC$ and $AD$. 3. Compare $BD$ and $BC$.

**462. Cor. 1.** *From a point outside a plane, a straight line can be drawn perpendicular to the plane.*




**Construction.** 1. Draw $DE$, any st. line in $MN$.

2. Draw $AF \perp$ to $DE$ at $F$, and $BF$, in $MN$, $\perp$ to $DE$ at $F$.

3. Draw $AB \perp$ to $BF$.

**Statement.** $AB \perp$ plane $MN$ from $A$.

**Proof.** 1. Draw $BE$.

2. $EF \perp$ the plane determined by $AF$ and $BF$. Why?

3. $\therefore BE \perp AB$. § 461

[Since $BF$, through the foot of $EF$, is $\perp$ to $AB$ in plane $ABF$.]

4. $\therefore AB \perp MN$.

[See step 3 of the Construction and of the Proof.]

**Note.**—*From a point outside a plane only one straight line can be drawn perpendicular to the plane.* If $AC$ and $AB$ were both $\perp$ to plane $MN$ from $A$, $\triangle ABC$ would have two right angles in it.

**463. Cor. 2.** *The perpendicular is the shortest segment that can be drawn from a point to a plane.*

**464.** The **Distance** from a point to a plane is the length of the perpendicular from the point to the plane.

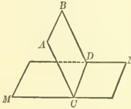
### PARALLEL LINES AND PLANES

**465.** A **straight line** is **parallel to a plane** if it does not meet the plane however far they are extended.

Two **planes** are **parallel** if they do not meet however far they are extended.

## Proposition VI. Theorem

**466.** *If a line outside a plane is parallel to a line of the plane, it is parallel to the plane.*




**Hypothesis.** $AB \parallel CD$.

Plane $MN$ contains $CD$ but not $AB$.

**Conclusion.** $AB \parallel$ plane $MN$.

**Proof.** 1. $AB$ and $CD$ lie in a plane $AD$. § 89

2. This plane intersects $MN$ in line $CD$. Why?

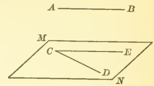
3. If $AB$ were to intersect $MN$, the point of intersection would be in plane $MN$ and also in plane $AD$, and therefore in $CD$.

4. Hence $AB$ would intersect $CD$.

5. But, $AB$ cannot meet $CD$. Why?

6. $\therefore$ $AB$ cannot meet $MN$ and hence $AB \parallel MN$.

**467. Cor. 1.** *Through a given straight line, a plane can be drawn parallel to any other straight line.*

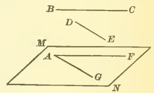



Prove a plane can be drawn through $CD \parallel AB$.

*Suggestion.* — Draw $CE \parallel AB$.

**Note.** — Discuss the solution when $AB \parallel CD$ and when $AB$ is not $\parallel CD$.

**468. Cor. 2.** *Through a given point, a plane can be drawn parallel to each of two given straight lines in space.*




Prove a plane can be drawn through point $A$, parallel to straight lines $BC$ and $DE$.

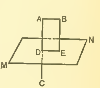
*Suggestion.* — Draw $AG \parallel DE$ and $AF \parallel BC$.

**Note.** — Discuss the solution when $BC \parallel DE$ and when $BC$ is not $\parallel DE$.

**Ex. 25.** Can more than one perpendicular be drawn to a line at a point of the line in a plane? in space?

**Ex. 26.** Through a given point outside a line, a plane can be drawn parallel to the line. How many such planes can be drawn?

**Ex. 27.** Prove that a straight line and a plane, both perpendicular to the same straight line, are parallel.




**Hyp.** $AB \perp AC$; plane $MN \perp AC$.

**Con.** $AB \parallel$ plane $MN$.

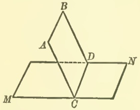
*Suggestion.* — Let the plane determined by $AB$ and $AC$ intersect $MN$ in line $DE$.

**Ex. 28.** Through a given point outside a plane, a line can be drawn parallel to the plane. Can more than one line be drawn parallel to the plane?

**Ex. 29.** Through a given point, a line can be drawn parallel to each of two intersecting planes.

## Proposition VII. Theorem

**469.** *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*




**Hypothesis.** $AB \parallel$ plane $MN$.
Plane $BC$, through $AB$, intersects $MN$ in $CD$.

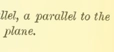
**Conclusion.** $AB \parallel CD$.

**Proof.** 1. $AB$ and $CD$ lie in the same plane $BC$.

2. $AB$ and $CD$ cannot intersect, for if they did, $AB$ would intersect plane $MN$, which is impossible.

3. $\therefore AB \parallel CD$. § 89

**470. Cor.** *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*




**Hyp.** $AB \parallel$ plane $MN$.
$C$ is any pt. in $MN$.
$CD \parallel AB$.

**Con.** $CD$ lies in $MN$.

**Proof.** 1. The plane determined by $AB$ and $C$ intersects $MN$ in a line $CE$, through $C$, parallel to $AB$. Why?

2. But $CD$, through $C$, $\parallel AB$. Why?

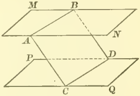
3. $\therefore CD$ and $CE$ coincide. Why?

4. $\therefore CD$ lies in plane $MN$.

**Ex. 30.** Prove that three non-concurrent straight lines, each of which intersects the other two, lie in a plane.

## Proposition VIII. Theorem

**471.** *If two parallel planes are cut by a third plane, the intersections are parallel.*




**Hypothesis.** Plane $MN \parallel$ plane $PQ$.
Plane $AD$ intersects $MN$ in $AB$ and $PQ$ in $CD$.

**Conclusion.** $AB \parallel CD$.

*Suggestion.* — Recall § 89.

**Ex. 31.** Prove that parallel segments between parallel planes are equal.

**Ex. 32.** If two planes are parallel, a line parallel to one of them through any point of the other lies in the other. (Fig. of Prop. VIII.)

*Suggestion.* — Given parallel planes $MN$ and $PQ$, and $AB$ through any point $A$ of $MN \parallel PQ$. Prove that $AB$ lies in $MN$. Through $AB$ pass a plane intersecting $MN$ in a line $AB'$ and $PQ$ in line $CD$. Consider the relation of lines $AB'$ and $CD$, and also of $AB$ and $CD$.

**Ex. 33.** From a line parallel to a plane, two parallels are drawn to the plane and terminated by the plane. Prove that the segments are equal.

### Review Exercises

**Ex. 34.** When is a line perpendicular to a plane?

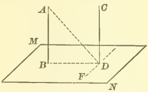
**Ex. 35.** Where do all lines lie which are perpendicular to a given line at a given point of the line?

**Ex. 36.** What is true about two lines in the same plane which are perpendicular to the same line?

**Ex. 37.** Line $AB$ is perpendicular to plane $MN$ at $A$. A line is drawn from $B$ meeting any line $CD$ of plane $MN$ at $E$. If line $BE$ is perpendicular to $CD$, prove $AE$ perpendicular to $CD$. (Fig. Prop. V.)

## Proposition IX. Theorem

**472.** *Two lines perpendicular to the same plane are parallel.*



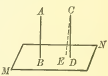

**Hypothesis.** $AB \perp MN$ at $B$; $CD \perp MN$ at $D$.

**Conclusion.** $AB \parallel CD$.

**Proof.** 1. Draw $AD$ from any pt. $A$ of $AB$. Draw $BD$.
In plane $MN$, draw $FD \perp BD$.

| | | |
|---|---|---|
| 2. | $CD \perp FD$. | Why? |
| 3. | $AD \perp FD$. | § 461 |
| 4. | $\therefore AD$, $BD$, and $CD$ lie in a plane $ABDC$. | § 458 |
| 5. | $\therefore AB$ and $CD$ lie in the same plane, $ABDC$. | |
| 6. | But $AB \perp BD$ and $CD \perp BD$. | Why? |
| 7. | $\therefore AB \parallel CD$. | § 97 |

**473. Cor. 1.** *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*

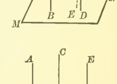



**Hyp.** $AB \parallel CD$.
$AB \perp$ plane $MN$.

**Con.** $CD \perp MN$.

*Suggestions.* — 1. Assume $CE \perp MN$.
2. Prove $CE \parallel AB$.

**474. Cor. 2.** *If each of two straight lines is parallel to a third straight line, they are parallel to each other.*




**Hyp.** $AB \parallel CD$; $EF \parallel CD$.

**Con.** $AB \parallel EF$.

*Suggestion.* — Let plane $MN$ be $\perp CD$.

**Ex. 38.** Through a given line which is parallel to a given plane, a number of planes are passed intersecting the given plane. Prove that the lines of intersection with the given plane are all parallel.

**Ex. 39.** If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

**Ex. 40.** Prove that a line parallel to a plane is everywhere equidistant from it.

**Ex. 41.** If two points are equidistant from a plane, and on the same side of it, they determine a line parallel to the plane.

**Ex. 42.** If two points lie on opposite sides of a plane and equidistant from it, the segment joining them is bisected by the plane.

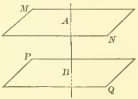
**Ex. 43.** If one of two parallel lines is parallel to a plane, the other is also, unless it lies in the plane.

*Suggestion.* — Through the line which is ∥ to the plane, pass a plane intersecting the given plane. Use § 466.

**Ex. 44.** Prove that the lines joining in order the mid-points of the sides of a quadrilateral in space form a parallelogram.

## Proposition X. Theorem

**475.** *Two planes perpendicular to the same straight line are parallel.*




**Hypothesis.** Planes $MN$ and $PQ$ are $\perp$ to $AB$.

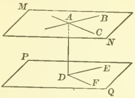
**Conclusion.** $MN \parallel PQ$.

*Suggestion.* — Prove it by the indirect method, using § 455.

**Ex. 45.** Are lines in space which are perpendicular to the same line necessarily parallel?

## Proposition XI. Theorem

**476.** *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*




**Hypothesis.** $AB \parallel$ plane $PQ$; $AC \parallel$ plane $PQ$.
$AB$ and $AC$ determine plane $MN$.

**Conclusion.** $MN \parallel PQ$.

**Proof.** 1. From $A$ draw $AD \perp PQ$.

2. $AC$ and $AD$ determine a plane which intersects $PQ$ in a line $DF$ parallel to $AC$. Why?

3. Similarly, $DE \parallel AB$.

4. $AD \perp DF$ and also $AD \perp DE$. Why?

5. $\therefore AD \perp AC$ and also $AD \perp AB$. Why?

Complete the proof, using § 475.

## Proposition XII. Theorem

**477.** *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*

**Hypothesis.** Plane $MN \parallel$ plane $PQ$. $AD \perp PQ$.

(Fig. § 476.)

**Conclusion.** $AD \perp MN$.

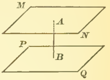
**Proof.** 1. Through $AD$ pass two planes intersecting $MN$ in $AB$ and $AC$, and $PQ$ in $DE$ and $DF$, respectively.

2. $\therefore AB \parallel DE$ and $AC \parallel DF$. Why?

3. $AD \perp DE$ and also $AD \perp DF$. Why?

Complete the proof.

**478.** The **distance between two parallel planes** is the length of the segment perpendicular to them and lying between them.



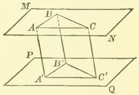

**479. Cor. 1.** *Two parallel planes are everywhere equally distant.*

**480. Cor. 2.** *Through a given point, a plane can be drawn parallel to a given plane.*

**Note.** — *Through a given point only one plane can be drawn parallel to a given plane.* For, if two planes through $A$ were parallel to $PQ$, each would be $\perp$ to $AB$ at $A$, and that is impossible.

## Proposition XIII. Theorem

**481.** *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*




**Hypothesis.** $\angle BAC$ is in plane $MN$; $\angle B'A'C'$ is in plane $PQ$.
$AB \parallel A'B'$ and extends in the same direction.
$AC \parallel A'C'$ and extends in the same direction.

**Conclusion.** $\angle BAC = \angle B'A'C'$; $MN \parallel PQ$.

**Proof.** 1. $AB \parallel$ plane $PQ$ and $AC \parallel PQ$. § 466

2. $\therefore MN \parallel PQ$.

3. Lay off $AB = A'B'$, and $AC = A'C'$; draw $BC$, $B'C'$, $AA'$, $BB'$, and $CC'$.

4. $ABB'A'$ is a $\square$, and $\therefore BB' = AA'$ and $BB' \parallel AA'$. Why?

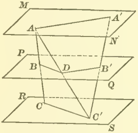
5. Similarly, $CC' = AA'$ and $CC' \parallel AA'$.

6. $\therefore BB'C'C$ is a $\square$. Prove it.

Complete the proof.

## Proposition XIV. Theorem

**482.** *If two straight lines are cut by three or more parallel planes, the corresponding segments are proportional.*




**Hypothesis.** Planes $MN$, $PQ$, and $RS$ are parallel.

$AC$ intersects the planes at $A$, $B$, and $C$ respectively.

$A'C'$ intersects the planes at $A'$, $B'$, and $C'$ respectively.

**Conclusion.** $\frac{AB}{BC}=\frac{A'B'}{B'C'}$.

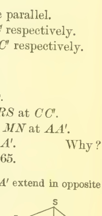
**Proof.** 1. Draw $AC'$ cutting $PQ$ at $D$.

2. Plane $CAC'$ intersects $PQ$ at $BD$ and $RS$ at $CC'$.
Plane $AC'A'$ intersects $PQ$ at $DB'$ and $MN$ at $AA'$.

3. $\therefore BD \parallel CC'$ and also $DB' \parallel AA'$. Why?
Complete the proof, using § 265.

**Ex. 46.** Discuss Prop. XIII: (*a*) if $BA$ and $B'A'$ extend in opposite directions and $CA$ and $C'A'$ in the same direction; (*b*) if both pairs extend in opposite directions from their vertices.

**Ex. 47.** If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

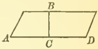



*Suggestion.* — Prove $\angle ABC = \angle DEF$.

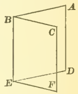
**Ex. 48.** If two planes are parallel to a third plane, they are parallel to each other. (§ 477 and § 475.)

## DIEDRAL ANGLES

**483.** It is evident that a straight line divides a plane into two parts, each indefinite in extent. A part of the plane, like $BCD$, is called a **Half-plane.** $BC$ is called the **Edge** of the half-plane.




**484.** A **Diedral Angle** is the figure formed by two half-planes which have a common edge.



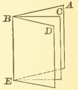

The common edge is the **Edge** and the two half-planes are the **Faces** of the diedral angle.

Thus, half-planes $BF$ and $BD$ form the diedral angle whose edge is $BE$, and whose faces are $FBE$ and $DBE$.

The diedral angle may be read:

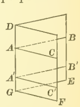
diedral $\angle BE$, or diedral $\angle ABEC$.

**485.** Two diedral angles are **Adjacent** when they have the same edge and a common face between them: as $\angle ABEC$ and $\angle CBED$.




Two diedral angles are **Vertical** when the faces of one are the extensions of the faces of the other.

**486.** A **Plane Angle** of a diedral angle is the angle formed by two straight lines one in each plane, drawn perpendicular to the edge at the same point.




Thus, if lines $AB$ and $AC$ be drawn in faces $DE$ and $DF$ respectively, of diedral angle $DG$, perpendicular to $DG$ at $A$, $\angle BAC$ is a plane angle of the diedral angle $DG$.

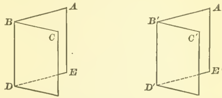
**487. Cor.** 1. *All plane angles of a given diedral angle are equal.*

**488. Cor.** 2. *A plane perpendicular to the edge of a diedral angle intersects the faces of the angle in lines which form the plane angle of the diedral angle.*

**489.** Two **diedral angles** are **equal** if they can be made to coincide.

## Proposition XV. Theorem

**490.** *Two diedral angles are equal if their plane angles are equal.*




**Hypothesis.** $\angle ABC$ and $\angle A'B'C'$ are the plane $\measuredangle$ of diedral $\measuredangle$ $BD$ and $B'D'$ respectively; $\angle ABC = \angle A'B'C'$.

**Conclusion.** $\angle BD = \angle B'D'$.

**Proof.** 1. Apply $\angle B'D'$ to $\angle BD$ so that $A'B'$ will coincide with $AB$ and $B'C'$ will coincide with $BC$.

2. $BD \perp$ plane $ABC$, and $B'D' \perp$ plane $A'B'C'$. Why?

3. $\therefore B'D'$ coincides with $BD$. Note, § 456

4. $\therefore A'D'$ coincides with $AD$, and $C'D'$ with $CD$. § 447, II

5. $\therefore \angle B'D' = \angle BD$. § 489

**491. Cor. 1.** *If two diedral angles are equal, their plane angles are equal.*

For the diedral angles can be made to coincide. Then a plane, perpendicular to the common edge, will intersect in each its plane angle, and evidently these plane angles coincide.

**492. Cor. 2.** *If two planes intersect, the vertical diedral angles are equal.*

*Suggestion.* — Compare their plane angles.

**Ex. 49.** Prove that a plane can be drawn bisecting a diedral angle.

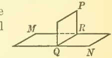
**493.** A **diedral angle** is **right, acute,** or **obtuse** according as its plane angle is right, acute, or obtuse. Two **diedral angles** are **supplementary** or **complementary** according as their plane angles are supplementary or complementary.

**Ex. 50.** If one plane meets another plane, the adjacent diedral angles formed are supplementary.

**Ex. 51.** If two parallel planes are cut by a third plane, the alternate-interior diedral angles are equal.

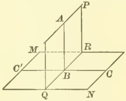
*Suggestion.* — Prove the plane ∠s of the alt.-int. diedral ∠s equal.

**494.** Two **planes** are **perpendicular** if the diedral angles formed are right diedral angles.




## Proposition XVI. Theorem

**495.** *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*




**Hypothesis.** $AB \perp$ plane $MN$.

$PQ$ is any plane through $AB$.

**Conclusion.** $PQ \perp MN$.

**Proof.** 1. Let $PQ$ and $MN$ intersect in line $QR$.

2. Draw $C'BC$ in $MN \perp QR$ at $B$.

3. $AB \perp QR$. Why?

4. $\therefore \angle ABC$ is the plane angle of the diedral $\angle PQRN$. Def.

5. But $\angle ABC$ is a rt. $\angle$. Why?

Complete the proof, recalling §§ 494 and 493.

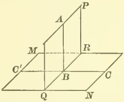
**Ex. 52.** (*a*) Prove that a plane can be drawn through a point perpendicular to a given plane.

(*b*) How many such planes can be drawn?

**Ex. 53.** Prove that a plane perpendicular to the edge of a diedral angle is perpendicular to the faces of the angle.

### Proposition XVII. Theorem

**496.** *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*




**Hypothesis.** Plane $PQ \perp$ plane $MN$. $PQ$ intersects $MN$ in $QR$. Line $AB$ in $PQ$ is $\perp QR$.

**Conclusion.** $AB \perp MN$.

**Proof.** 1. Draw $C'BC$ in plane $MN \perp QR$.

2. $\therefore \angle ABC$ is the plane $\angle$ of diedral $\angle PQRN$. Why?

3. But $\angle PQRN$ is a rt. diedral $\angle$. Why?

Complete the proof.

**497. Cor.** 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

**Hyp.** Plane $PQ \perp$ plane $MN$, intersecting it in $QR$. $AB$, drawn from any pt. $B$ of $QR$, is $\perp$ to $MN$.

**Con.** $AB$ lies in $PQ$.

*Suggestions.* — 1. A line in $PQ \perp QR$ at $B$ is $\perp MN$. Why?
2. Prove that it and $AB$ coincide.

**498. Cor.** 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other lies in the other.*

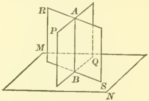
**Hyp.** Plane $PQ \perp$ plane $MN$, intersecting it in $QR$; $AB$, drawn from any pt. $A$ of $PQ$, is $\perp$ to $MN$.

**Con.** $AB$ must lie in $PQ$.

*Suggestions.* — In $PQ$ draw a $\perp$ to $QR$ from $A$. Prove that $AB$ must coincide with this perpendicular.

## Proposition XVIII. Theorem

**499.** *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*




**Hypothesis.** Plane $MN \perp$ planes $RS$ and $PQ$.
$RS$ intersects $PQ$ in line $AB$.

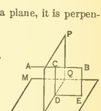
**Conclusion.** $MN \perp AB$.

*Suggestion.* — Assume a line $\perp MN$ from $A$. Where will this line lie? (§ 498.)

**Ex. 54.** Are two planes which are perpendicular to the same plane necessarily parallel?

**Ex. 55.** If a plane is perpendicular to a line of a plane, it is perpendicular to the plane.

**Ex. 56.** If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

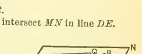



**Hyp.** $AB \parallel$ plane $MN$.
Plane $PR \perp AB$ at $C$.

**Con.** $PR \perp MN$.

*Suggestions.* — 1. Draw line $CD$ in $PR \perp QR$.
2. Let the plane determined by $CB$ and $CD$ intersect $MN$ in line $DE$.
3. Prove $CD \perp MN$.

**Ex. 57.** Prove that any point in the plane through the bisector of an angle and perpendicular to the plane of the angle is equidistant from the sides of the angle.

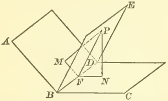



*Suggestions.* — 1. Draw $PR \perp OC$, $RT \perp OA$, and $RS \perp OB$. Draw $PT$ and $PS$.

2. Prove $PR \perp MN$ (§ 496), $PT \perp OA$ (§ 461), $PS \perp OB$.

## Proposition XIX. Theorem

**500.** *Every point in the plane bisecting a diedral angle is equidistant from the faces of the angle.*




**Hypothesis.** Plane $BE$ bisects diedral $\angle ABDC$.
$P$ is any point in plane $BE$.
$PM \perp$ plane $AD$; $PN \perp$ plane $DC$.

**Conclusion.** $PM = PN$.

**Proof.** 1. The plane determined by $PM$ and $PN$ intersects planes $AD$, $BE$, and $CD$ in lines $FM$, $FP$, and $FN$ respectively.

2. Plane $PMFN \perp AD$ and also $\perp CD$. Why?

3. $\therefore PMFN \perp BD$. Why?

4. $\therefore$ ∠s $PFM$ and $PFN$ are the plane ∠s of diedral ∠s $ABDE$ and $CBDE$. Why?

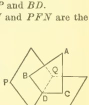
5. $\therefore \angle PFM = \angle PFN$. Why?

Complete the proof.

**501. Cor.** *Any point within a diedral angle and equidistant from its faces lies in the plane bisecting the diedral angle.*

*Suggestions.* — 1. Let $BE$ be the plane determined by $P$ and $BD$.
2. Prove that $BE$ bisects $\angle ABDC$ by proving ∠s $PFM$ and $PFN$ are the plane angles of the diedral angles and are equal.

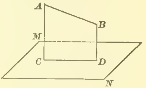
**Ex. 58.** If perpendiculars are drawn to the faces of a diedral angle from any point within the angle, they lie in a plane perpendicular to the edge of the diedral angle and form an angle which is the supplement of the plane angle of the diedral angle.




*Suggestion.* — What is the sum of the angles of a plane quadrilateral?

## Proposition XX. Theorem

**502.** *Through a given straight line not perpendicular to a given plane, one and only one plane can be drawn perpendicular to the given plane.*




**Hypothesis.** $AB$ is not $\perp$ to plane $MN$.

**Conclusion.** A plane can be drawn through $AB \perp MN$, and only one.

**Proof.** 1. Draw $AC \perp MN$, from point $A$.

2. $AC$ and $AB$ determine a plane $\perp MN$. Why?

3. If a second plane through $AB$ were $\perp MN$, their intersection $AB$ would also be $\perp MN$. Why?

4. But $AB$ is not $\perp MN$.

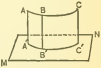
5. $\therefore$ Only one plane can be drawn through $AB \perp MN$.

**Note 1.** — If $AB$ lies in $MN$, the theorem is still true.

**Note 2.** — If $AB \perp MN$, an infinity of planes can be drawn through $AB \perp MN$ (§ 495).

**503.** The **projection of a point on a plane** is the foot of the perpendicular drawn from the point to the plane.

The **projection of a given line on a plane** is the line which contains the projections of all the points of the given line.




Thus, $A'B'C'$ is the projection of $ABC$ on $MN$.

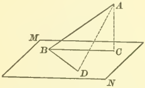
**504. Cor.** *The projection of a straight line on a plane is a straight line.* (Fig. § 502.)

*Suggestions.* — 1. Through $AB$ pass a plane $AD \perp MN$.

2. Prove that the feet of the $\perp$s to $MN$ from $AB$ lie in $CD$. (§ 498.)

## Proposition XXI. Theorem

**505.** *The acute angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*




**Hypothesis.** $BC$ is the projection of $AB$ on plane $MN$.
$BD$ is any other line in $MN$ through $B$.

**Conclusion.** $\angle ABC < \angle ABD$.

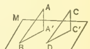
*Suggestions.* — 1. Take $BD = BC$. 2. Compare $AD$ and $AC$. 3. Then compare $\measuredangle ABC$ and $ABD$, recalling § 167.

**506.** The **angle between a line and a plane** is the acute angle made by the line with its projection on the plane. This angle is called the **Inclination** of the line to the plane.

**Ex. 59.** If two equal segments are drawn to a plane from a point outside the plane, they make equal angles with the plane.

**Ex. 60.** If two parallels meet a plane, they make equal angles with it.

*Suggestion.* — Given $AB \parallel CD$; $AA' \perp MN$, and $CC' \perp MN$. Prove $\angle ABA' = \angle CDC'$.




**Ex. 61.** Prove that a straight line and its projection upon a plane lie in a plane which is perpendicular to the given plane.

**Ex. 62.** If a straight line-segment is parallel to a plane, it is parallel to its projection upon the plane, and is equal to it.

**Ex. 63.** If two parallel lines are oblique to a plane, their projections upon the plane are parallel. (§ 481 and § 471.)

**Ex. 64.** Prove that the ratio of two parallel line-segments is the same as the ratio of their projections upon a given plane.

**Ex. 65.** Can the projection upon a plane of a curved or broken line be a straight line?

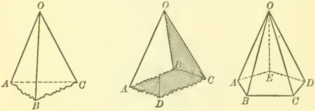
**Ex. 66.** If the projection upon a plane of a given figure is a straight line, then the figure lies in a plane.

**Ex. 67.** Let the projection upon a given plane $M$ of a segment $l$ be denoted by $l'$. What is the relation between $l$ and $l'$ if:
(a) $l \perp M$? (b) $l \parallel M$? (c) $l$ and $M$ form an angle of 45°?

**Note.** — Supplementary Exercises 1–17, p. 454, can be studied now.

## POLYEDRAL ANGLES

**507.** The figures below represent **polyedral angles.**




$\angle$ *O-ABC* $\angle$ *O-ABCD* $\angle$ *O-ABCDE*

Notice that each is formed of portions of three or more intersecting planes; these planes are the **Faces** of the polyedral angle. The faces intersect in one common point; this is the **Vertex** of the polyedral angle. Each face has two edges which pass through the vertex of the angle; these are the **Edges** of the polyedral angle. On each face, the two edges form an angle, whose vertex is also the vertex of the polyedral angle; these angles are the **Face Angles** of the polyedral angle. Each pair of consecutive faces intersect in an edge, forming a diedral angle; these diedral angles are the **Diedral Angles** of the polyedral angle.

The edges and the faces are unlimited in extent. It is convenient to indicate the polygon which results if a plane is drawn, not through the vertex, but intersecting all the faces; this polygon aids in picturing the number of faces of the polyedral angle.

**508.** A **Triedral Angle** is a polyedral angle having three faces.

A **Tetraedral Angle** is a polyedral angle having four faces.

**Ex. 68.** If a polyedral angle has 4 faces, how many vertices, edges, face angles, and diedral angles does it have?

**Ex. 69.** Name the edges, face angles, and the diedral angles of triedral angle $O\text{-}ABC$. (Fig. § 507.)

**509.** Two **polyedral angles** are **vertical** if the edges of one are the prolongations of the edges of the other.

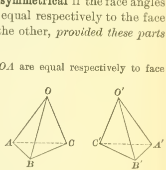
**510.** Two **polyedral angles** are **congruent** if they can be made to coincide.

**Ex. 70.** (*a*) Construct two triedral angles which have the face angles of one equal respectively to the face angles of the other and in the same order. Determine whether they can he made to coincide.

(*b*) Construct a third triedral angle whose face angles are equal to those of the triedrals of part (*a*), but arrange them in order opposite to that in part (*a*). (See Fig. § 511.) Can this triedral angle be made to coincide with either of those in part (*a*)?

**511.** Two **polyedral angles** are **symmetrical** if the face angles and the diedral angles of one are equal respectively to the face angles and the diedral angles of the other, *provided these parts occur in opposite orders.*

Thus, if face $\measuredangle AOB$, $BOC$, and $COA$ are equal respectively to face $\measuredangle A'O'B'$, $B'O'C'$, and $C'O'A'$, and diedral $\measuredangle OA$, $OB$, and $OC$ to diedral $\measuredangle O'A'$, $O'B'$, and $O'C'$, triedral $\measuredangle O\text{-}ABC$ and $O'\text{-}A'B'C'$ are symmetrical, since the parts in $\angle O'\text{-}A'B'C'$ occur in opposite order to the equal parts of $\angle O\text{-}ABC$; that is, to pass from $OA$ to $OB$ to $OC$, one moves from left to right, whereas, to pass from $O'A'$ to $O'B'$ to $O'C'$, one moves from right to left.




It is evident that, in general, two symmetrical polyedrals cannot be placed so that their faces will coincide.

## Proposition XXII. Theorem

**512.** *Two vertical polyedral angles are symmetrical.*

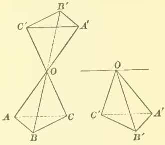



Fig. 1 Fig. 2

**Hypothesis.** $O\text{–}ABC$ and $O\text{–}A'B'C'$ are vertical triedral angles. (Fig. 1.)

**Conclusion.** $\measuredangle$ $O\text{–}ABC$ and $O\text{–}A'B'C'$ are symmetrical.

**Proof.** 1. Face $\measuredangle$ $AOB$, $BOC$, etc. equal respectively face $\measuredangle$ $A'OB'$, $B'OC'$, etc. Why?

2. Diedral $\measuredangle$ $BOAC$ and $B'OA'C'$ are vertical; for $AOB$ and $A'OB'$ are parts of the same plane, as also are $AOC$ and $A'OC'$. Similarly, $\measuredangle$ $OB$ and $OB'$ are vertical, etc.

3. $\therefore$ Diedral $\measuredangle$ $OA$, $OB$, etc. equal respectively diedral angles $OA'$, $OB'$, etc. § 492

4. The parts of $O\text{–}ABC$ occur in opposite order to the equal parts of $\angle$ $O\text{–}A'B'C'$.

This may be understood by moving $O\text{–}A'B'C'$ parallel to itself to the right, and then revolving it about an axis through $O$ (as shown in Fig. 2) until face $OA'C'$ comes into the same plane as before. $OB'$ is then in front of plane $C'OA'$ instead of back of that plane as in Fig. 1. Now, in Fig. 1, to pass from $AO$ to $OB$ to $OC$, one moves from left to right; in Fig. 2 to pass from $OA'$ to $OB'$ to $OC'$, one moves from right to left.

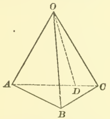
5. $\therefore$ $\measuredangle$ $O\text{–}ABC$ and $O\text{–}A'B'C'$ are symmetrical. § 511

**Note.** — The theorem may be proved for any two vertical polyedral angles in the same manner.

## Proposition XXIII. Theorem

**513**. *The sum of any two face angles of a triedral angle is greater than the third.*

**Note**. — The theorem requires proof only in the case when the third face angle is greater than either of the others.



**Hypothesis**. In triedral $\angle O\text{-}ABC$

$\angle AOC > \angle AOB$, and also $\angle AOC > \angle BOC$

**Conclusion**. $\angle AOC < \angle AOB + \angle BOC.$

**Proof**. 1. In face $AOC$, draw $OD = OB$, making $\angle AOD = \angle AOB$.

2. Through $B$ and $D$ pass any plane cutting the faces of the triedral $\angle$ in $AB$, $BC$, and $CA$, respectively.

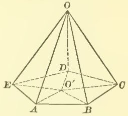
| | | |
|---|---|---|
| 3. | $\triangle AOB \cong \triangle AOD.$ | Prove it. |
| 4. | $\therefore AB = AD.$ | Why? |
| | In $\triangle ABC$ | |
| 5. | $AB + BC > AD + DC.$ | Why? |
| 6. | $\therefore BC > DC.$ | § 158, Ax. 18 |
| 7. | $\therefore$ in $\triangle BOC$ and $\triangle COD$ | |
| | $\angle BOC > \angle COD.$ | § 167 |
| 8. | $\therefore \angle AOB + \angle BOC > \angle AOD + \angle COD.$ | Why? |
| 9. | $\therefore \angle AOB + \angle BOC > \angle AOC.$ | |

**Ex. 71**. Prove that any face angle of a polyedral angle is less than the sum of the remaining face angles.

*Suggestion*. — Divide the polyedral $\angle$ into triedral $\angle$s by passing planes through any lateral edge, and apply § 513.

## Proposition XXIV. Theorem

**514.** *The sum of the face angles of any convex polyedral angle is less than four right angles.*



**Hypothesis.** $O\text{-}ABCDE$ is any convex polyedral $\angle$.

**Conclusion.** $\angle AOB + \angle BOC +$ etc. $< 4$ rt. $\measuredangle$.

**Proof.** 1. Pass a plane cutting the faces in the polygon $ABCDE$.

Let $O'$ be any point within $ABCDE$, and draw $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

2. Then, in triedral $\angle A\text{-}EOB$,

$\angle OAE + \angle OAB > \angle EAO' + \angle O'AB$. § 513

3. Similarly $\angle OBA + \angle OBC > \angle ABO' + \angle O'BC$; etc.

4. Adding these inequalities, the sum of the base $\measuredangle$ of the $\triangle$ whose common vertex is $O$ is *greater than* the sum of the base $\measuredangle$ of the $\triangle$ whose common vertex is $O'$. § 158, Ax. 19

5. The sum of *all* the $\measuredangle$ of the $\triangle$ with vertex $O$ equals the sum of all the $\measuredangle$ of the $\triangle$ with vertex $O'$. Why?

6. $\therefore$ the sum of the $\measuredangle$ at $O$ is *less than* the sum of the $\measuredangle$ at $O'$. § 158, Ax. 20

7. $\therefore$ the sum of the $\measuredangle$ at $O < 4$ rt. $\measuredangle$. Why?

**Note.** — The pupil's understanding of this theorem will be increased if a pasteboard model of the figure of § 514 is at hand when this theorem is first studied.

Notice that the inequality in step 2 *does not mean* that $\angle EAO > \angle EAO'$ and $\angle OAB > \angle O'AB$; rather, the sum of $\angle EAO$ and $\angle OAB$ is greater than the face angle $EAB$ of triedral $\angle A\text{-}EOB$.

## Proposition XXV. Theorem

**515.** *If two triedral angles have the face angles of one equal respectively to the face angles of the other, their homologous diedral angles are equal.*

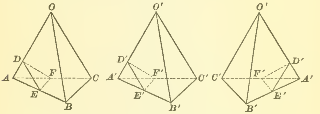


FIG. 1 FIG. 2 FIG. 3

**Hypothesis.** In triedral $\angle s$ $O\text{–}ABC$ and $O'\text{–}A'B'C'$ $\angle AOB = \angle A'O'B'$; $\angle BOC = \angle B'O'C'$; $\angle COA = \angle C'O'A'$.

**Conclusion.** Diedral $\angle OA =$ diedral $\angle O'A'$; etc.

**Proof.** 1. Lay off $OA$, $OB$, $OC$, $O'A'$, $O'B'$, and $O'C'$ all of the same length, and draw $AB$, $BC$, $CA$, $A'B'$, $B'C'$, and $C'A'$.

2. $\therefore \triangle OAB \cong \triangle O'A'B'$ and $AB = A'B'$. Prove it.

3. Similarly $BC = B'C'$ and $AC = A'C'$.

4. Also $\triangle ABC \cong \triangle A'B'C'$ and $\angle BAC = \angle B'A'C'$. Why?

5. On $OA$ and $O'A'$, take $AD = A'D'$; draw $DE$ in face $OAB \perp OA$, $D'E'$ in $A'O'B' \perp A'O'$, $DF$ in $AOC \perp AO$, $D'F'$ in $A'O'C' \perp A'O'$, $EF$ in face $ABC$, and $E'F'$ in face $A'B'C'$.

6. Then $\triangle ADE \cong \triangle A'D'E'$. Prove it.

$\therefore AE = A'E'$ and $DE = D'E'$.

7. Also $AF = A'F'$ and $DF = D'F'$. Prove it.

8. Then $\triangle AEF \cong \triangle A'E'F'$ and $EF = E'F'$. Prove it.

9. Then $\triangle DEF \cong \triangle D'E'F'$. Prove it.

10. $\therefore \angle EDF = \angle E'D'F'$. Why?

11. $\therefore$ diedral $\angle AO =$ diedral $\angle A'O'$. Why?

**Note.** — The above proof holds for Fig. 3 as well as for Fig. 2. In Figs. 1 and 2, the equal parts occur in the same order, and in Figs. 1 and 3 in the reverse order.

**516. Cor.** *If two triedral angles have the face angles of one equal respectively to the face angles of the other,*

1. *They are congruent if the equal parts occur in the same order.*

2. *They are symmetrical if the equal parts occur in the reverse order.*

## SUPPLEMENTARY TOPICS

The following topics, theorems, and exercises of Book VI are supplementary. *Each group is independent of each of the others.* None of this material is required in the main parts of subsequent Books.

Group *A*. — Analogy between Triedral Angles and Triangles.

Group *B*. — Loci in Space.

Group *C*. — Consists of two supplementary theorems usually given in texts.

## Group A. Analogy between Triedral Angles and Triangles

**517.** The **analogy between triangles and triedral angles** is very striking. Many propositions of plane geometry about triangles may be changed into propositions about triedral angles by substituting for the word *angle* of the former *diedral angle*, and for the word *side* the words *face angle*.

**Ex. 72.** Two triedral angles are congruent when a face angle and the adjacent diedral angles of one are equal respectively to a face angle and the adjacent diedral angles of the other, if the parts are arranged in the same order.

*Suggestion.* — Prove by superposition.

**Note.** — If the parts of one are arranged in reverse order to the parts of the other, the triedral angles are symmetrical.

**Ex. 73.** Two triedral angles are congruent if two face angles and the included diedral angle of one are equal respectively to two face angles and the included diedral angle of the other, if the parts are arranged in the same order.

**Ex. 74.** If two face angles of a triedral angle are equal, the opposite diedral angles are equal.

*Suggestion.* — Recall the proof of § 69.

**Ex. 75.** An *exterior diedral angle* of a triedral angle is greater than either remote interior diedral angle.

*Suggestion.* — Model the proof after that in § 87.

**Ex. 76.** — If two triedral angles have a face angle, the opposite diedral angle and another diedral angle of one equal respectively to the corresponding parts of the other, they are congruent if the parts are in the same order.

*Suggestion.* — Superpose the equal face angles, so that the equal diedral angles adjacent to the faces superposed also coincide. Prove that the faces opposite these diedral angles also coincide by an indirect proof, based upon Exercise 75.

**Note.** — If the parts are in reverse order, the figures are symmetrical.

**Ex. 77.** State and prove the converse of Ex. 74.

*Suggestion.* — The proof is based upon Ex. 76, Note.

## Group B. Loci in Space

**518.** The following more general definition of locus will be employed in solid geometry. The **locus of points** satisfying a given condition consists of all points which satisfy the condition, and of no other points.

The points which constitute a locus may form one (or more) lines, or one or more surfaces.

To prove that a particular assumed locus is actually the locus satisfying a given condition, or conditions, prove either (*a*) and (*b*) below or else (*a*) and (*c*).

(*a*) *Every point of the locus satisfies the conditions.*

(*b*) *Every point not of the locus does not satisfy the conditions.*

(*c*) *Every point which does satisfy the conditions lies in the locus.*

**Ex. 78.** As a consequence of § 457 and § 459, what is the locus of points equidistant from the extremities of a line?

**Ex. 79.** As a consequence of § 500 and § 501, what is the locus of points equidistant from two intersecting planes?

**Ex. 80.** What is the locus of points in space at a given distance from a given plane?

**Ex. 81.** What is the locus of points in space equally distant from two parallel planes?

**Ex. 82.** What is the locus of points in space equidistant from the points of a given circle?

**Ex. 83.** What is the locus of points in space equidistant from the vertices of a given triangle?

**519. Intersection Loci.** — When two conditions are imposed upon a point in space, each condition determines a locus for the point, and the desired point lies in the intersection of the loci. It is often inadvisable to attempt to draw the two loci, for that demands considerable skill in drawing. The following form of solution brings out all the mathematical value of such a problem quite as well, if not better, than if a figure were drawn.

ILLUSTRATIVE PROBLEM. — *What is the locus of points in space at a given distance from a given plane and equidistant from two given points?*

**Solution.** 1. The locus of points at a given distance from a given plane consists of two planes parallel to the given plane and at the given distance from it. Call this Locus 1.

2. The locus of points equidistant from two given points is the plane perpendicular to and bisecting the segment between the two points. Call this Locus 2.

3. The desired locus is the intersection of Locus 1 and Locus 2.

**Discussion.** 1. Generally the plane of Locus 2 will intersect the planes of Locus 1 in two straight lines.

2. Locus 2 may be parallel to the planes of Locus 1. In this case, there will not be any points satifying the given conditions.

3. Locus 2 may coincide with one of the planes of Locus 1. In this case, the plane common to Loci 1 and 2 will be the desired locus.

**Ex. 84.** What is the locus of points in a given plane equidistant from two parallel planes?

**Ex. 85.** What is the locus of points in a given plane at a given distance from another given plane?

**Ex. 86.** What is the locus of points in a given plane equidistant from two given points not in the plane?

**Ex. 87.** What is the locus of points in a given plane equidistant from two intersecting planes?

**Ex. 88.** What is the locus of points equidistant from two given points and also equidistant from two parallel planes?

**Ex. 89.** What is the locus of points equidistant from two given points and also equidistant from two intersecting planes?

**Ex. 90.** What is the locus of points equidistant from two parallel planes, and also equidistant from two intersecting planes?

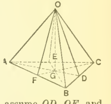
**Ex. 91.** What is the locus of points equidistant from two intersecting planes, and also at a given distance from a given plane?

**Ex. 92.** Find all points which are at a given distance from a given plane, equidistant from two other parallel planes, and equidistant from two given points.

**Ex. 93.** Find all points which are equidistant from two given intersecting planes, equidistant from two parallel planes, and equidistant from two given points.

**Ex. 94.** Prove that the three planes bisecting the diedral angles of a triedral angle meet in a common straight line.

*Suggestion.*—Planes $OAD$ and $OBE$ intersect in line $OG$. Prove $OG$ is in plane $OCF$.




**Ex. 95.** Prove that the three planes determined by the edges of a triedral angle and the bisectors of the opposite face angles intersect in a line.

*Suggestions.*—1. In a figure like that for Ex. 94, assume $OD$, $OF$, and $OE$ bisect the angles $BOC$, $BOA$, and $AOC$, respectively.

2. Take $OA = OB = OC$, and draw $AB$, $BC$, and $AC$.
3. Prove $AD$, $CF$, and $EB$ are concurrent.
4. Prove planes $AOD$, $BOE$, and $COF$ meet in a line.

**Ex. 96.** Prove that the locus of points equidistant from the sides of an angle is the plane through the bisector of the angle, and perpendicular to the plane of the angle.

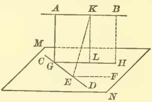
*Suggestion.*—Recall § 460 and Ex. 57, Book VI.

**Ex. 97.** Prove that the planes through the bisectors of the face angles of a triedral angle, perpendicular to the planes of the faces, meet in a line, whose points are equidistant from the edges of the triedral angle.

## Group C. Supplementary Theorems

### Proposition XXVI. Theorem

**520.** *Two straight lines not in the same plane have one common perpendicular, and only one; and this segment is the shortest segment that can be drawn between them.*




**Hypothesis.** $AB$ and $CD$ do not lie in the same plane.

**Conclusion.** One and only one common $\perp$ to $AB$ and $CD$ can be drawn; also, this $\perp$ is the shortest segment between $AB$ and $CD$.

**Proof.** 1. Through $CD$ draw plane $MN \parallel AB$. § 467

2. Through $AB$, draw plane $AH \perp$ plane $MN$, intersecting $MN$ in line $GH$. § 502

3. $\therefore GH \parallel AB$. Why?

4. $\therefore GH$ intersects $CD$ at a point $G$.

[If $CD \parallel GH$, $CD$ would be $\parallel$ to $AB$, which is impossible.]

5. In plane $AH$, draw $AG \perp GH$ at $G$.

6. Then $AG \perp AB$ and also to $CD$. Prove it.

7. Assume $KE$ also $\perp$ to both $AB$ and $CD$.

8. Draw $EF \parallel AB$, and $KL$ in plane $AH \perp GH$.

9. $EF$ is in $MN$. § 470

10. $EK \perp EF$. Why?

11. $\therefore EK \perp MN$. Why?

12. But this is impossible for $KL \perp MN$. § 496

13. Hence $AG$ is the only common $\perp$ to $AB$ and $CD$. Why?

14. $EK > KL$. Why?

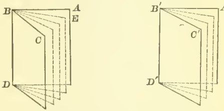
15. $\therefore EK > AG$. Why?

16. $\therefore AG$ is the shortest segment from $AB$ to $CD$.

### Proposition XXVII. Theorem

**521.** *Two diedral angles have the same ratio as their plane angles.*

Case I. *When the plane angles are commensurable.*




**Hypothesis.** $\measuredangle ABC$ and $A'B'C'$, the plane $\measuredangle$ of diedral $\measuredangle$ $ABDC$ and $A'B'D'C'$ respectively, are commensurable.

**Conclusion.** $$\frac{\angle A'B'D'C'}{\angle ABCD} = \frac{\angle A'B'C'}{\angle ABC}.$$

**Proof.** 1. Let $\angle ABE$, a common measure of $\angle ABC$ and $\angle A'B'C'$, be contained 4 times in $\angle ABC$ and 3 times in $\angle A'B'C'$.

2. Then $$\frac{\angle A'B'C'}{\angle ABC} = \frac{3}{4}.$$

3. Passing planes through the edges $BD$ and $B'D'$, and the division lines of $\angle ABC$ and $\angle A'B'C'$, respectively, diedral $\angle ABCD$ is divided into 4 parts, and $\angle A'B'D'C'$ into three parts, all of which are equal. Why?

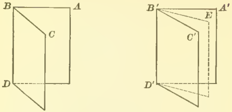
4. $$\therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{3}{4}.$$

5. $$\therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}.$$ Why?

Case II. *When the plane angles are incommensurable.*

**Hypothesis.** $\angle ABC$ and $\angle A'B'C'$, plane $\measuredangle$ of diedral $\measuredangle$ $ABDC$ and $A'B'D'C'$, respectively, are incommensurable.

**Conclusion.** $$\frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}.$$




**Proof.** 1. Let $\angle ABC$ be divided into any number of equal parts, and let one of these parts be applied to $\angle A'B'C'$ as unit of measure.

Since $\angle ABC$ and $\angle A'B'C'$ are incommensurable, the unit will not be contained exactly in $\angle A'B'C'$.

A certain number of equal parts will extend from $A'B'$ to $B'E$, leaving the remainder $\angle EB'C'$ less than the unit of measure.

2. Pass a plane through $B'D'$ and $B'E$. Then

$$\frac{\angle A'B'D'E}{\angle ABDC} = \frac{\angle A'B'E}{\angle ABC}. \qquad \text{Case I}$$

3. Now let the number of subdivisions of $\angle ABC$ be indefinitely increased; then the unit of measure will be indefinitely decreased, and consequently the remainder $\angle EB'C'$ will approach the limit 0. § 401

4. $$\therefore \frac{\angle A'B'D'E}{\angle ABDC} \doteq \frac{\angle A'B'D'C'}{\angle ABDC}. \qquad \text{§ 403, (a)}$$

["$\doteq$," means "approaches the limit."]

5. Also $$\frac{\angle A'B'E}{\angle ABC} \doteq \frac{\angle A'B'C'}{\angle ABC}. \qquad \text{§ 403, (a)}$$

6. $$\therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}. \qquad \text{§ 403, (b)}$$

# BOOK VII

## POLYEDRA

**522. Surfaces.** So far only *plane surfaces* have been considered.

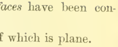
A **Curved Surface** is a surface no part of which is plane.

The surface of a spherical object is a familiar example of a curved surface.

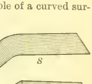
It is evident that there may be surfaces of which part is plane and part is curved; as surface $S$.



Also there are surfaces consisting of two or more parts each of which is plane; as surface $T$.



**523. Closed Surfaces.** Let a plane $ABC$ intersect the faces of triedral angle $O\text{–}XYZ$, and consider the surface consisting of triangles $OAB$, $OAC$, $OBC$, and $ABC$, and the portions of planes within them. This surface incloses a *finite* portion of space. Such a surface is a *closed surface.* A closed surface is such that the intersection of it made by *every* intersecting plane is a closed line.



**524.** A **closed surface** is **convex** if the intersection with it of every intersecting plane is a convex closed line.

It will be assumed that all closed surfaces considered in this text are convex.

**525.** A **Solid** is the finite portion of space inclosed by a closed surface. The surface is called the *surface of the solid,* and is said to bound the solid.

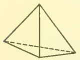
In the remaining part of solid geometry a detailed study is made of certain common solids.

**526.** A **Polyedron** is a solid bounded by portions of planes, called the **Faces** of the polyedron. The faces intersect in straight lines, called the **Edges** of the polyedron. The edges intersect in points, called the **Vertices** of the polyedron. The straight line joining any two vertices of the polyedron which do not lie in the same face is a **Diagonal** of the polyedron.

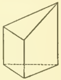
**527.** The least number of planes that can form a polyedral angle is three. Then the least number of planes that can form a polyedron is four.

A polyedron of four faces is a **Tetraedron**; one of six faces is a **Hexaedron**; one of eight faces is an **Octaedron**; one of twelve faces is a **Dodecaedron**; and one of twenty is an **Icosaedron.**

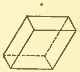
The cube is a familiar hexaedron.



TETRAEDRON



PENTAEDRON



HEXAEDRON

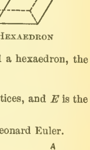
**Ex. 1.** Verify for a tetraedron, a pentaedron, and a hexaedron, the formula

$$F + V = E + 2,$$

where $F$ is the number of faces, $V$ is the number of vertices, and $E$ is the number of edges.

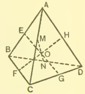
**Note.** — This theorem is due to the mathematician Leonard Euler.

**Ex. 2.** If $E$, $F$, $G$, and $H$ are the mid-points of edges $BD$, $BC$, $AC$, and $AD$, respectively, of tetraedron $ABCD$, prove $EFGH$ a parallelogram.

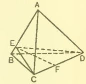


**Ex. 3.** If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges $AB$ and $CD$ is a parallelogram.

**Note.** — Remember that by § 468 a plane *can be* drawn parallel to each of two straight lines in space.




**Ex. 4.** The lines joining, by pairs, the mid-points of opposite edges of any tetraedron, intersect in a common point.



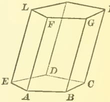

**Ex. 5.** In tetraedron $ABCD$, a plane is drawn through edge $CD$ perpendicular to $AB$, intersecting faces $ABC$ and $ABD$ in $CE$ and $ED$, respectively. If the bisector of $\angle CED$ meets $CD$ at $F$, prove

$$CF : DF = \text{area } ABC : \text{area } ABD.$$

*Suggestion.* — Recall § 270.

### PRISMS AND PARALLELOPIPEDS

**528.** A **Prism** is a polyedron, two of whose faces lie in parallel planes, and whose remaining faces, in order, intersect in parallel lines. The parallel faces are the **Bases**; the other faces are the **Lateral Faces**; the edges which are not sides of the bases are the **Lateral Edges**; the perpendicular between the bases is the **Altitude**; the sum of the areas of the lateral faces is the **Lateral Area**.




If a plane is perpendicular to the lateral edges of a prism, its intersection with the prism is called a **Right Section** of the prism.

**529.** The following **Important Facts about a Prism** should be proved by the pupil:

I. *The lateral faces of a prism are inclosed by parallelograms.*

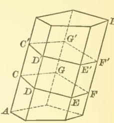
Prove $BCHG$ is a ▱, using § 471. Fig. § 528

II. *The lateral edges of a prism are parallel and equal.*

III. *The bases of a prism are inclosed by congruent polygons.*

*Suggestion.* — Recall § 481.

IV. *Sections of the lateral surface of a prism made by two parallel planes cutting all the lateral edges are congruent polygons.*

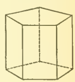



*Suggestion.* — Let planes $CF$ and $C'F'$ be parallel.

Prove $CDEFG \cong C'D'E'F'G'$.

V. *A section of a prism made by a plane parallel to the base is congruent to the base.*

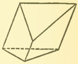
**530. Kinds of Prisms.** A prism is *triangular, quadrangular,* etc., according as its base is *triangular, quadrangular,* etc.



A **Right Prism** is a prism whose lateral edges are perpendicular to its bases.

An **Oblique Prism** is a prism whose lateral edges are not perpendicular to its bases.

A **Regular Prism** is a right prism whose base is inclosed by a regular polygon.



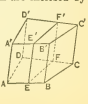
A **Truncated Prism** is that portion of a prism bounded by the base and a plane not parallel to the base, cutting all the lateral edges.

**Ex. 6.** Prove that every pair of lateral edges of a prism determines a plane which is parallel to each of the other lateral edges of the prism.

**Ex. 7.** Prove that the lateral edges of a right prism are equal to the altitude.

**Ex. 8.** Prove that the lateral faces of a right prism are inclosed by rectangles.

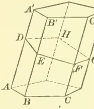
**Ex. 9.** Prove that the section of a prism made by a plane parallel to a lateral edge is inclosed by a parallelogram.




*Suggestion.*—Let plane $EF'$ be $\parallel AA'$, a lateral edge of prism $AC'$.

## Proposition I. Theorem

**531.** *The lateral area of a prism equals the perimeter of a right section multiplied by the length of a lateral edge.*




**Hypothesis.** $DEFGH$ is a rt. section of prism $AC'$.

$P =$ perimeter of $DEFGH$; $L =$ length of $AA'$;

$S =$ the lateral area.

**Conclusion.** $S = LP$.

**Proof.** 1. Area of $\square\ AB' = L \times DE$. Why?

2. Similarly, area of $\square\ BC' = L \times EF$. Why?

Complete the proof.

**532. Cor.** *The lateral area of a right prism equals the perimeter of the base multiplied by the length of the altitude.*

**Ex. 10.** Find the lateral area of a regular hexagonal prism, each side of whose base is 3 and whose altitude is 9.

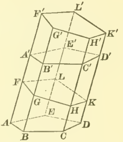
**Ex. 11.** There are upon a porch six columns having the form of regular octagonal prisms. If the side of the base is 4 inches and the altitude of the column is 7 feet, find the total of the lateral areas of the columns.

**533.** The **Volume of a Solid** is a number which indicates the measure of that solid in terms of a unit of solid measure; it is the ratio of the solid to the unit of solid measure.

**534.** Two **solids** are **equal** if they have equal volumes. Evidently two congruent solids are equal; likewise two solids which can be divided into parts which are respectively congruent are equal.

## Proposition II. Theorem

**535.** *An oblique prism equals a right prism which has for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*




**Hypothesis.** $FK'$ is a right prism. Its base $FK$ is a right section of oblique prism $AD'$.

Its altitude $FF' = AA'$, a lateral edge of $AD'$.

**Conclusion.** $FK' = AD'$.

**Proof.** 1. $ABCDE \cong A'B'C'D'E'$,

and $FGHKL \cong F'G'H'K'L'$. § 529, IV

2. $AF = A'F'$; $BG = B'G'$; $CH = C'H'$; etc. Prove it.

3. Slide polyedron $AK$ upward, letting $AF$, $BG$, etc., move along lines $AF'$, $BG'$, etc., until $ABCDE$ coincides with $A'B'C'D'E'$.

4. Then $F$, $G$, $H$, etc., fall upon $F'$, $G'$, $H'$, etc. Step 2

5. $\therefore$ $FGHKL$ will fall upon $F'G'H'K'L'$.

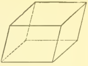
6. $\therefore$ polyedron $AK \cong$ polyedron $A'K'$.

7. $\therefore$ prism $AD' =$ prism $FK'$. § 534

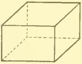
**536. Cor.** *Two right prisms are congruent, and hence equal, when they have congruent bases and equal altitudes.*

For the bases can be made to coincide. Then the lateral edges of one will coincide with the homologous edges of the other by § 530 and Note, § 456. Then the upper bases must coincide.

**537.** A **Parallelopiped** is a prism whose base is inclosed by a parallelogram. As a consequence, all the faces of a parallelopiped are inclosed by parallelograms.



A **Right Parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases.



A **Rectangular Parallelopiped** is a right parallelopiped whose base is inclosed by a rectangle. Consequently all the faces are inclosed by rectangles. (See Ex. 8, p. 349.)

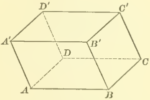
A **Cube** is a rectangular parallelopiped whose six faces are inclosed by squares.

**Ex. 12.** How many faces of a right parallelopiped are rectangles?

**Ex. 13.** Is a cube a prism?

## Proposition III. Theorem

**538.** *The opposite lateral faces of a parallelopiped are congruent and parallel.*




**Hypothesis.** $AC$ and $A'C'$ are the bases of parallelopiped $AC'$.

**Conclusion.** Faces $AB'$ and $DC'$ are congruent and $\parallel$.

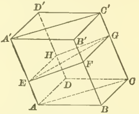
*Suggestions.* — 1. To prove $AB' \cong DC'$, prove them mutually equiangular and mutually equilateral.

2. To prove $AB' \parallel DC'$, recall § 481.

**539. Cor.** *Any pair of opposite faces of a parallelopiped may be taken as its bases.*

## Proposition IV. Theorem

**540.** *The plane through two diagonally opposite edges of a parallelopiped divides it into two equal triangular prisms.*




**Hypothesis.** Plane $AC'$ passes through edges $AA'$ and $CC'$ of parallelopiped $A'C$.

**Conclusion.** Prism $A'\text{-}ABC =$ prism $A'\text{-}ACD$.

**Proof.** 1. Let $EFGH$ be a right section of the parallelopiped, intersecting plane $AA'C'C$ in $EG$.

2. $EFGH$ is a $\square$. Prove it.

3. $\therefore \triangle EFG \cong \triangle EGH$. Why?

4. Prism $A'\text{-}ABC =$ a right prism with base $EFG$ and altitude $AA'$.

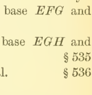
Prism $A'\text{-}ACD =$ a right prism with base $EGH$ and altitude $AA'$. § 535

5. But these right prisms are equal. § 536

6. $\therefore A'\text{-}ABC = A'\text{-}ACD$.

**Ex. 14.** Prove that the diagonals of a rectangular parallelopiped are equal.

**Ex. 15.** Prove that the diagonals of a parallelopiped bisect each other.




*Suggestion.* — Prove that each of the other diagonals bisects $BG$.

**Note.** — The point of intersection of the diagonals of a parallelopiped is called the **center of the parallelopiped.**

**Ex. 16.** Prove that any line drawn through the center of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

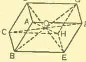
**Ex. 17.** Prove that the line joining the center of a parallelopiped to the center of any face is parallel to any edge of the parallelopiped which intersects that face, and is equal to one half of it.

**Ex. 18.** Prove that the centers of two opposite faces of a parallelopiped and the center of the parallelopiped are collinear.

*Suggestion.* — Recall Ex. 17 and § 90.

**Ex. 19.** If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

**Plan.** — Prove $BCHE$ is a ▱.




*Suggestion.* — $DE$ and $AH$ determine a plane (why?) which intersects planes $DF$ and $BH$ in lines $AD$ and $HE$, respectively. Compare $AD$ and $HE$; also compare $AD$ and $BC$. Then compare $BC$ and $HE$.

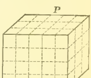
**Ex. 20.** Recall that the diagonals of a square are equal, bisect each other, and are perpendicular to each other.

What questions about the diagonals of a cube do these facts suggest? Prove the answers to your questions.

**541.** The **Dimensions** of a rectangular parallelopiped are the *lengths* of its three edges which meet at any vertex.

The **Volume of a rectangular parallelopiped** is readily determined when the three dimensions are multiples of the linear unit of measure.

P



Thus, if the dimensions of $P$ are 5 units, 4 units, and 3 units, respectively, the solid can be divided into 60 unit cubes. In this case 60, the number which expresses the area, is the product of 3, 4, and 5, the three dimensions.

The following three propositions prove that this is the correct formula for determining the volume of any rectangular parallelopiped. Before taking up these propositions, certain necessary new ideas are introduced.*

* The class may have studied limits in Plane Geometry (§ 401 and following). In that case the following two paragraphs constitute a review of those paragraphs of the Plane Geometry.

**542.** *a.* A **Variable** is a number which assumes different values during a particular discussion.

Thus a number $x$ which assumes successively the values $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ is a *decreasing variable*, assuming ultimately values which differ by as little as we please from zero.

A number $y$ which assumes successively the values $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \cdots$ is an *increasing variable*, assuming ultimately values which differ by as little as we please from 2.

*b.* A **Constant** is a number which has a fixed value throughout a particular discussion.

*c.* A **Limit of a Variable** is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number. We say that a *variable approaches its limit.* If a variable has a limit, it has only one limit. The symbol $\doteq$ means "approaches the limit."

Thus, $x$ above $\doteq 0$, and $y \doteq 2$.

**543. Limits Theorems.** It can be proved that:

*a.* If a variable $x$ approaches a finite limit $l$, then $cx$, where $c$ is a constant, approaches the limit $cl$. § 403, *a*

*b.* If two variables are constantly equal and each approaches a finite limit, then their limits are equal. § 403, *b*

*c.* If a variable $x$ approaches a finite limit $a$, and a variable $y$ approaches a limit $b$, then:

$(x \pm y)$ approaches the limit $a \pm b$;

$xy$ approaches the limit $a \cdot b$;

and $\frac{x}{y}$ approaches the limit $\frac{a}{b}$, provided $b$ is not zero.

**Ex. 21.** Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 8, 9, and 12.

**Ex. 22.** Prove that the square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its edges.

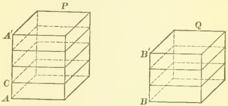
**Ex. 23.** Determine the length of the diagonal of a cube whose edge is of length $s$.

**Note.** — Supplementary Exercises 18–22, p. 456, can be studied now.

### Proposition V. Theorem

**544.** *Two rectangular parallelopipeds having congruent bases have the same ratio as their altitudes.*

Case I. *When the altitudes are commensurable.*




**Hypothesis.** $P$ and $Q$ are rectangular parallelopipeds with congruent bases, and commensurable altitudes $AA'$ and $BB'$.

**Conclusion.** $$\frac{Q}{P}=\frac{BB'}{AA'}.$$

**Proof.** 1. Let $AC$, a common measure of $AA'$ and $BB'$, be contained 4 times in $AA'$ and 3 times in $BB'$.

$$\therefore \frac{BB'}{AA'}=\frac{3}{4}.$$

2. Through the points of division of $AA'$ and $BB'$ draw planes $\perp AA'$ and $BB'$, respectively.

3. Then $P$ is divided into 4, and $Q$ is divided into 3, parts which are all congruent. § 536

4. $$\therefore \frac{Q}{P}=\frac{3}{4}.$$

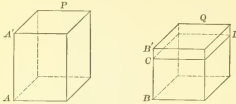
5. $$\therefore \frac{Q}{P}=\frac{BB'}{AA'}.$$ Why?

Case II (Fig. p. 357). *When the altitudes are incommensurable.**

**Hypothesis.** $P$ and $Q$ are rectangular parallelopipeds with congruent bases and incommensurable altitudes $AA'$ and $BB'$.

* This proof may be omitted if desired.

**Conclusion.** $$\frac{Q}{P}=\frac{BB'}{AA'}.$$




**Proof.** 1. Divide $AA'$ into any number of equal parts, and apply one of these parts to $BB'$ as unit of measure. Since $AA'$ and $BB'$ are incommensurable, a certain number of these parts will extend from $B$ to $C$, leaving a remainder $CB'$ less than the unit of measure.

2. Draw plane $CD \perp BB'$, and let parallelopiped $BD$ be denoted by $Q'$.

Then $$\frac{Q'}{P}=\frac{BC}{AA'}.$$ Case I

[Since $AA'$ and $BC$ are commensurable.]

3. Let the number of subdivisions in $AA'$ be indefinitely increased. The length of each subdivision will then diminish indefinitely, and hence $CB'$ will approach the limit *O*. § 542, *c*

Also $\frac{Q'}{P}$ will approach the limit $\frac{Q}{P}$, § 543, *a*

and $\frac{BC}{AA'}$ will approach the limit $\frac{BB'}{AA'}$. § 543, *a*

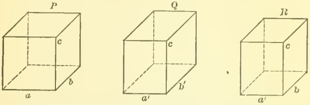
4. $$\therefore \frac{Q}{P}=\frac{BB'}{AA'}.$$ § 543, *b*

**545. Cor.** *If two rectangular parallelopipeds have two dimensions of one equal respectively to two dimensions of the other, their volumes have the same ratio as their third dimensions.*

**Ex. 24.** Compare the volumes of two rooms having the same floor space if their heights are 8′ 6″ and 9′ respectively.

## Proposition VI. Theorem

**546.** *Two rectangular parallelopipeds having equal altitudes have the same ratio as their bases.*




**Hypothesis.** Rectangular parallelopipeds $P$ and $Q$ have equal altitudes $c$ and bases with dimensions $a$, $b$, and $a'$, $b'$, respectively.

**Conclusion.** $$\frac{P}{Q}=\frac{a \times b}{a' \times b'}.$$

**Proof.** 1. Let $R$ be a rectangular parallelopiped with dimensions $c$, $a'$, and $b$.

2. Then $$\frac{P}{R}=\frac{a}{a'},$$ § 545

and $$\frac{R}{Q}=\frac{b}{b'}.$$ Why?

3. Then, multiplying the equations of step 2,

$$\frac{P}{Q}=\frac{ab}{a'b'}.$$ Ax. 5, § 51

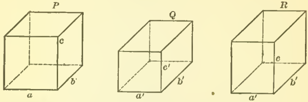
**Note.** — Two rectangular parallelopipeds having a dimension of one equal to one dimension of the other, have the same ratio as the products of their other two dimensions.

**Ex. 25.** Two rectangular parallelopipeds, with equal altitudes, have the dimensions of their bases 6 and 14, and 7 and 9, respectively. Find the ratio of their volumes.

**Ex. 26.** Compare each of the following rectangular parallelopipeds with each of the others: $R$, having dimensions 5, 7, and 9; $S$, having dimensions 9, 5, and 4; $T$, having dimensions 4, 6, and 7.

## Proposition VII. Theorem

**547.** *Two rectangular parallelopipeds have the same ratio as the products of their three dimensions.*




**Hypothesis.** $P$ and $Q$ are rectangular parallelopipeds having dimensions $a$, $b$, $c$, and $a'$, $b'$, $c'$, respectively.

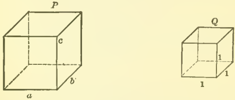
**Conclusion.** $$\frac{P}{Q}=\frac{abc}{a'b'c'}.$$

*Suggestions.* — 1. Let $R$ be a rectangular parallelopiped with dimensions $a'$, $b'$, and $c$.

2. Find $\frac{P}{R}$ and $\frac{R}{Q}$ by § 546 and § 545. Then multiply $\frac{P}{R}$ by $\frac{R}{Q}$.

## Proposition VIII. Theorem

**548.** *If the unit of solid measure is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped equals the product of its three dimensions.*




**Hypothesis.** $a$, $b$, and $c$ are the dimensions of rectangular parallelopiped $P$, and $Q$ is the unit of measure.

**Conclusion.** Volume of $P=a\times b\times c$.

**Proof.** 1. Volume of $P=\frac{P}{Q}$. § 533

Complete the proof, applying § 547.

**549. Cor. 1.** *The volume of a cube is equal to the cube of its edge.*

**550. Cor. 2.** *The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

**Note.** — Corollaries 1 and 2 are expressed in their commonly abbreviated form. Expressed more accurately, the second would be "the *volume* of a rectangular parallelopiped is equal to the product of the *area* of the base and the *length* of the altitude." The brief form of statement will be employed in the remaining theorems of solid geometry.

Remember that *volume* of a solid means the *number* of cubic units in it. In all succeeding theorems relating to volumes, it is understood that the *unit of solid* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit.

**Ex. 27.** Find the ratio of the volumes of two rectangular parallelopipeds whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

**Ex. 28.** Find the volume and the area of the entire surface of a cube whose edge is 4 in.

**Ex. 29.** (*a*) If the edge of a cube is $e$, express by formulæ the total area, the volume, and the length of a diagonal. (*b*) Using the proper formula, determine the edge when the diagonal is 12. (*c*) Using the proper formula, determine the edge when the total area is 150 sq. in.

**Ex. 30.** Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equal to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

**Ex. 31.** What must be the height of a tank having the form of a rectangular parallelopiped whose base has the dimensions 5 ft. and 8 ft., in order that the tank will contain 1800 gal. of water when the water rises to within one foot of the top? (One cu. ft. of water $= 7\frac{1}{2}$ gal. approximately.)

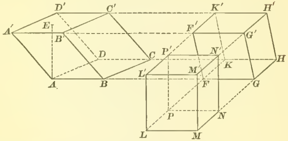
**Ex. 32.** How many barrels of water will run into a cistern during a $\frac{1}{2}$ in. fall of rain from the roof of a barn whose total roof area is 800 sq. ft. (One cu. ft. of water $= 7\frac{1}{2}$ gal.)

**Ex. 33.** (*a*) How many cubic yards of concrete are required for the foundation walls of a house 25 ft. $\times$ 35 ft., if the walls are 10 in. thick and are 8 ft. high? (*b*) How many bags of cement are required if the mixture contains 4 bags of cement to one yard of gravel?

**Note.** — Supplementary Exercises 23–28, p. 456, can be studied now.

## Proposition IX. Theorem

**551.** *The volume of any parallelopiped is equal to the product of its base and altitude.*




**Hypothesis.** $AC'$ is an oblique parallelopiped.

Let the length of altitude $AE$ be $H$, the area of base $ABCD$ be $B$, and the volume of $AC'$ be $V$.

**Conclusion.** $V = BH$.

**Proof.** 1. Extend edges $AB$, $A'B'$, $D'C'$, and $DC$.

On $AB$ extended, take $FG = AB$.

Draw planes $FK'$ and $GH' \perp FG$, forming right parallelopiped $FH'$.

2. $\therefore$ prism $FH' =$ prism $AC'$. § 535

3. Extend edges $HG$, $H'G'$, $K'F$, and $KF$.

On $HG$ extended, take $NM = HG$; draw planes $NP'$ and $ML' \perp NM$, forming rectangular parallelopiped $LN'$.

4. $\therefore$ prism $L'N =$ prism $FH'$. Why?

5. $\therefore$ prism $L'N =$ prism $AC'$. Why?

6. But volume $L'N = LMNP \times MM'$.

7. $\therefore$ volume $AC' = LMNP \times MM'$.

8. But the length of $MM' = H$

and area $LMNP =$ area $ABCD = B$. Note, p. 362

9. $\therefore$ vol. $AC' = BH$.

**Note.** — The student's understanding of this theorem will be increased greatly if a model of the above figure is at hand.

**Note 1.** — *Proof that $LN'$* (step 3, § 551) *is a rectangular parallelopiped.*

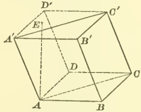
1. Since $FG \perp$ plane $GH'$, $\therefore$ plane $LH \perp$ plane $MH'$. § 495
2. Since $MM' \perp MN$, $\therefore MM' \perp$ plane $LH$. § 496
3. $\therefore \angle LMM' =$ a rt. $\angle$.
4. $\therefore LM'$ is a rectangle. § 141
5. $\therefore LN'$ is a rectangular parallelopiped. § 537

**Note 2.** — *Proof that $LMNP = ABCD$.*

$LMNP = FGHK$, since ▱s having equal bases and equal altitudes are equal. Similarly $FGHK = ABCD$, and $\therefore LMNP = ABCD$.

## Proposition X. Theorem

**552.** *The volume of a triangular prism is equal to the product of its base and altitude.*




**Hypothesis.** $C'\text{-}ABC$ is a triangular prism.

Length of altitude $AE = H$; area of $\triangle ABC = B$; volume of $C'\text{-}ABC = V$.

**Conclusion.** $V = BH$.

*Suggestions.* — 1. Consider the parallelopiped $D'\text{-}ABCD$, having its edges parallel to $AB$, $BC$, and $BB'$ respectively.

2. Compare volume of $C'\text{-}ABC$ with that of $D'\text{-}ABCD$.

3. Express the volume of $D'\text{-}ABCD$, and then of $C'\text{-}ABC$.

**Note.** — At this point, the pupil should memorize the following formulæ if they are not already known.

1. Area of a $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, § 335
where the sides are $a$, $b$, and $c$, and $s = \frac{1}{2}(a+b+c)$.

2. Area of an equilateral $\triangle$ of side $s = \frac{s^2\sqrt{3}}{4}$. (Ex. 29, p. 199, Book IV.)

3. Altitude of an equilateral $\triangle$ of side $s = \frac{s\sqrt{3}}{2}$.

**Ex. 34.** Find the volume of a regular triangular prism the side of whose base is 5 and whose altitude is 10.

**Ex. 35.** Derive a formula for the volume of a regular triangular prism the side of whose base is $s$ and whose altitude is $h$.

**Ex. 36.** Find the lateral area and volume of a right triangular prism, having the sides of its base 4, 7, and 9, respectively, and the altitude 8.

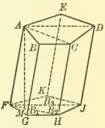
*Suggestion.* — To determine the area of the base, use the first formula in the note of § 552.

**Ex. 37.** Prove that the volume of a right triangular prism is equal to the product of the area of any face and one half the altitude to that face.

**Ex. 38.** The volume of a triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

## Proposition XI. Theorem

**553.** *The volume of any prism is equal to the product of its base and altitude.*




**Hypothesis.** Let $H =$ the length of altitude $AM$, $B =$ the area of base $FGHJK$, and $V =$ the volume of prism $AJ$.

**Conclusion.** $V = BH.$

*Suggestions.* — 1. Through edge $AF$ and diagonals $FH$ and $FJ$ of the base, pass planes $AFHC$ and $AFJD$ dividing prism $P$ into triangular prisms $P_1$, $P_2$, and $P_3$, whose base areas are $B_1$, $B_2$, and $B_3$, respectively, and whose common altitude is of length $H$.

2. Express the volumes of $P_1$, $P_2$, and $P_3$. Add the results and simplify, thus obtaining an expression for the volume $P$.

**554. Cor. 1.** *Two prisms having equal bases and equal altitudes are equal.*

*Suggestion.* — Let prisms $P$ and $P'$ have equal bases $B$ and $B'$ respectively, and equal altitudes $H$ and $H'$. Prove $P = P'$.

**555. Cor. 2.** *Two prisms having equal altitudes have the same ratio as their bases.*

**556. Cor. 3.** *Two prisms having equal bases have the same ratio as their altitudes.*

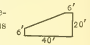
**557. Cor. 4.** *Two prisms have the same ratio as the products of their bases and altitudes.*

**Ex. 39.** Find the volume of a regular hexagonal prism the side of whose base is 4 in. and whose altitude is 9 in.

**Ex. 40.** Express by formulæ the total area and the volume of a regular hexagonal prism whose base edge is $e$ and whose height is $h$.

**Ex. 41.** A contractor agreed to dig a cellar at 35¢ per cubic yard. The lot was located upon a hillside so that the depth of the cellar at the back was 9 ft. and in front 5 ft. If the cellar was 38 ft. from the front to back, and was 25 ft. wide, how much did the contractor receive?

**Ex. 42.** How many cubic yards of concrete are required for a retaining wall 2 ft. thick whose dimensions are indicated on the adjoining figure?



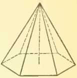

Note. — Supplementary Exercises 29–33, p. 457, can be studied now.

## PYRAMIDS

**558.** A **Pyramid** is a polyedron bounded by three or more triangular faces which have a common vertex, and one other plane face, the **Base**, which intersects each of the triangular faces.

The common vertex of the triangular faces is the **Vertex** of the pyramid; the triangular faces are the **Lateral Faces**; the edges terminating at the vertex are the **Lateral Edges**; the sum of the areas of the lateral faces is the **Lateral Area**; the perpendicular from the vertex to the plane of the base is the **Altitude**.

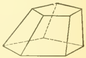
**559.** A pyramid is called *triangular, quadrangular,* etc., according as its base is *triangular, quadrangular,* etc.



A **Regular Pyramid** is a pyramid whose base is inclosed by a regular polygon, and whose vertex lies in the perpendicular to the base at the center of the base.

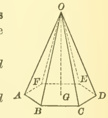
A **Truncated Pyramid** is the part of a pyramid included between its base and a plane cutting all the lateral edges.

The base of the pyramid and the section of the cutting plane are called the *bases of the truncated pyramid.*



**560.** A **Frustum of a Pyramid** is a truncated pyramid whose bases are parallel.

The **Altitude** of a frustum is the perpendicular between the planes of the bases.

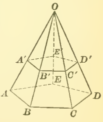



**561.** The following **important facts about pyramids** should be proved by the pupil:

I. *The lateral edges of a regular pyramid are equal.*

II. *The lateral faces of a regular pyramid are inclosed by congruent isosceles triangles.*

III. *The lateral faces of a frustum of any pyramid are inclosed by trapezoids.*




IV. *The lateral faces of a frustum of a regular pyramid are inclosed by congruent trapezoids.*

*Suggestion.* — Superpose $\triangle OAB$ on $\triangle OBC$. Prove $ABB'A' \cong BCC'B'$.

V. *The lateral edges of a frustum of a regular pyramid are equal.*

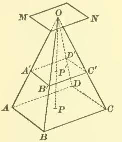
**Note.** — It will be assumed that the boundary of the base of the pyramid is a convex polygon.

## Proposition XII. Theorem

**562.** *If a pyramid is cut by a plane parallel to the base:*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*




**Hypothesis.** Plane $A'C'$, parallel to the base of pyramid $O\text{-}ABCD$, intersects faces $OAB$, $OBC$, $OCD$, and $ODA$ in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude $OP$ at $P'$.

**Conclusion.** I. $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OD'}{OD} = \frac{OP'}{OP}$.

II. $A'B'C'D' \sim ABCD$.

**Proof of I.** 1. Through $O$, pass plane $MN \parallel ABCD$.

2. $\therefore \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = \frac{OP'}{OP}$. § 482

**Proof of II.** 1. $\angle A'B'C' = \angle ABC$, $\angle B'C'D' = \angle BCD$, etc. Prove it. § 481

2. $\frac{A'B'}{AB} = \frac{OA'}{OA}$, $\frac{B'C'}{BC} = \frac{OB'}{OB}$, etc. Why?

3. $\therefore \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA}$. Why?

4. $\therefore A'B'C'D' \sim ABCD$. Why?

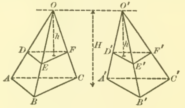
**563. Cor. 1.** *The area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

**Proof.** 1. $\dfrac{\text{Area } A'B'C'D'}{\text{Area } ABCD} = \dfrac{\overline{A'B'}^2}{\overline{AB}^2}$. § 344

2. But $\dfrac{A'B'}{AB} = \dfrac{OA'}{OA} = \dfrac{OP'}{OP}$.

3. $\therefore \dfrac{\text{Area } A'B'C'D'}{\text{Area } ABCD} = \dfrac{\overline{OP'}^2}{\overline{OP}^2}$.

**564. Cor. 2.** *If two pyramids have equal altitudes and equal bases, sections parallel to the bases at equal distances from the vertices are equal.*



**Hypothesis.** Pyramids $O\text{-}ABC$ and $O'\text{-}A'B'C'$ have the common altitude $H$, and equal bases, $ABC$ and $A'B'C'$.

$DEF$ and $D'E'F'$ are sections of the pyramids parallel to the bases at the distance $h$ from the vertices $O$ and $O'$, respectively.

**Conclusion.** $DEF = D'E'F'$.

**Proof.** 1. $\dfrac{\text{Area } DEF}{\text{Area } ABC} = \dfrac{h^2}{H^2}$ and $\dfrac{\text{Area } D'E'F'}{\text{Area } A'B'C'} = \dfrac{h^2}{H^2}$. § 563

Complete the proof.

**Ex. 43.** Prove that the areas of sections of a pyramid made by planes parallel to the base have the same ratio as the squares of the distances to the planes from the vertex.

**Ex. 44.** The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.?

**Ex. 45.** What part of the area of the base of a pyramid is the area of a section made by a plane which is parallel to the base and bisects the altitude?

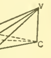
**Note.** — Supplementary Exercises 34–37, p. 457, can be studied now.

**565.** The **slant height of a regular pyramid** is the altitude of any lateral face. (See II, § 561.)

The **slant height of a frustum** of a regular pyramid is the altitude of any lateral face. (See III, § 561.)

**Ex. 46.** Prove that the perimeter of the mid-section of a frustum of a pyramid is one half the sum of the perimeters of the bases.

**Ex. 47.** What is the slant height of a regular quadrangular pyramid whose altitude is 12 and the side of whose base is 4?



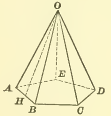

*Suggestions.* — 1. Let $ABV$ be one face, $VC$ be the altitude, and $C$ the center of the base.

2. Determine the length of $AC$, and of $DC$; then of $VD$.

**Ex. 48.** What is the slant height of a regular hexagonal pyramid whose altitude is 10 and whose base edge is 4?

## Proposition XIII. Theorem

**566.** *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one half its slant height.*




**Hypothesis.** $O$–$ABCDE$ is a regular pyramid.

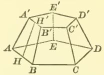
$P$ = perimeter of its base; $L$ = length of its slant height; $S$ = the lateral area.

**Conclusion.** $S = \frac{1}{2} PL.$

**Proof. 1.** Area of $\triangle OAB = \frac{1}{2} L \times AB.$ Why?

Complete the proof.

**567. Cor.** *The lateral area of the frustum of a regular pyramid is equal to one half the sum of the perimeters of the bases multiplied by the slant height.*




**Hypothesis.** $AD'$ is a frustum of a regular pyramid.

$L =$ the length of the slant height; $p$ and $P =$ the perimeters of the upper and lower bases respectively; and $S =$ the lateral area.

**Conclusion.** $S = \frac{1}{2} L(p + P)$.

**Proof.** 1. Area of $ABB'A' = \frac{1}{2} L(AB + A'B')$. Why?

Complete the proof.

**Ex. 49.** What is the slant height of a regular triangular pyramid whose altitude is 10 and whose base edge is 4?

*Suggestion.* — Recall § 172.

**Ex. 50.** Express the lateral area of a regular pyramid in terms of the length of the slant height and the perimeter of the section midway between the base and the vertex.

**Ex. 51.** Prove the lateral surface of any pyramid greater than its base, when the perpendicular from the vertex to the base falls within the base.

*Suggestion.* — From the foot of the altitude draw lines to the vertices of the base; each △ formed has a smaller altitude than the corresponding lateral face.

**Ex. 52.** Determine the lateral area of each of the pyramids in Exercises 47 to 49.

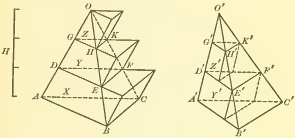
**Ex. 53.** In each of the Exercises 47 to 49 pass a plane parallel to the base at a distance of 5 in. from the vertex. Determine the lateral areas of the resulting frustums.

**Ex. 54.** Determine the total areas of the pyramids of Exercises 47 to 49.

**Ex. 55.** The edges of the bases of a frustum of a regular square pyramid are 5 in. and 10 in. respectively, and the altitude is 6 in. Determine the slant height and then the lateral area.

## Proposition XIV. Theorem

**568.** *Two triangular pyramids having equal altitudes and equal bases are equal.*




**Hypothesis.** $O\text{–}ABC$ and $O'\text{–}A'B'C'$ have equal altitudes and equal bases $ABC$ and $A'B'C'$.

**Conclusion.** $O\text{–}ABC = O'\text{–}A'B'C'$.

**Proof.** 1. Place the pyramids with their bases in the same plane, and let $H$ represent their common altitude.

Divide $H$ into 3 equal parts.

Through the points of division pass planes ∥ to the plane of the bases, cutting $O\text{–}ABC$ in sections $DEF$ and $GHK$, and $O'\text{–}A'B'C'$ in sections $D'E'F'$ and $G'H'K'$.

2. $\therefore DEF = D'E'F'$
and $GHK = G'H'K'$. § 564

3. With $ABC$, $DEF$, and $GHK$ as *lower* bases, construct prisms $X$, $Y$, and $Z$ with their lateral edges equal and ∥ to $AD$; with $D'E'F'$ and $G'H'K'$ as *upper* bases, construct prisms $Y'$ and $Z'$, with their lateral edges equal and ∥ to $A'D'$.

4. $\therefore$ prism $Y =$ prism $Y'$
and prism $Z =$ prism $Z'$. Why?

5. Hence the sum of the prisms *circumscribed* about $O\text{–}ABC$ exceeds the sum of the prisms *inscribed* in $O\text{–}A'B'C'$ by prism $X$.

6. Evidently, $O\text{–}ABC < X + Y + Z$,

and $O\text{–}ABC > Y' + Z'$.

Likewise $O'\text{–}A'B'C' < X + Y + Z$ and $> Y' + Z'$.

7. $\therefore$ $O\text{–}ABC$ and $O'\text{–}A'B'C'$ differ by less than the difference between the sum of the circumscribed prisms and the sum of the inscribed prisms;

*i.e.* $O\text{–}ABC$ and $O'\text{–}A'B'C'$ differ by less than the lower prism $X$.

8. By increasing indefinitely the number of subdivisions of $H$, the volume of $X$ can be made less than any assigned number, however small.

9. Suppose now that the volume of $O\text{–}ABC$ and $O'\text{–}A'B'C'$ differ by any amount $k$.

Since $X >$ the difference between $O\text{–}ABC$ and $O'\text{–}A'B'C'$, then $X$ would be $> k$.

10. But this contradicts step 8.

11. $\therefore$ $O'\text{–}A'B'C'$ and $O\text{–}ABC$ cannot differ at all;

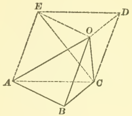
*i.e.* $O'\text{–}A'B'C' = O\text{–}ABC$.

**Note.** — An interesting and instructive exercise at this point is that of proving the equality of two triangles which have equal bases and altitudes, by a proof like that given for Proposition XIV. In fact, it aids in understanding Proposition XIV if the exercise proposed is studied before taking up § 568.

The two triangles are compared with two sets of parallelograms, one inscribed in one triangle, the other circumscribed about the other triangle. The resulting figures are like the triangles $AOB$ and $A'O'B'$ of the figure of § 568.

## Proposition XV. Theorem

**569.** *The volume of a triangular pyramid is equal to one third the product of its base and altitude.*




**Hypothesis.** $H =$ the length of the altitude; $B =$ the area of the base; and $V =$ the volume of pyramid $O\text{–}ABC$.

**Conclusion.** $V = \frac{1}{3} HB$.

**Proof.** 1. Let $EOD\text{–}ABC$ be the triangular prism having base $ABC$, and its lateral edges equal and parallel to $OB$.

2. Prism $EOD\text{–}ABC$ is composed of pyramid $O\text{–}ABC$ and pyramid $O\text{–}ACDE$.

3. Divide $O\text{–}ACDE$ into two triangular pyramids, $O\text{–}ACE$ and $O\text{–}CDE$ by passing a plane through $E$, $O$, and $C$.

4. In pyramids $O\text{–}ACE$ and $O\text{–}ECD$:

The altitudes are common. Why?

Base $ACE =$ base $ECD$. Why?

$\therefore O\text{–}ACE = O\text{–}ECD$. Why?

5. Pyramid $O\text{–}ECD$ is the same as pyramid $C\text{–}EOD$.

6. In pyramids $O\text{–}ABC$ and $C\text{–}EOD$:

The altitudes are equal. Why?

Base $ABC =$ base $EOD$. Why?

$\therefore O\text{–}ABC = C\text{–}EOD$. Why?

7. $\therefore O\text{–}ABC = O\text{–}ECD = O\text{–}ACE$.

8. $\therefore O\text{–}ABC = \frac{1}{3}$ prism $EOD\text{–}ABC$.

9. The altitude of prism $EOD\text{–}ABC = H$, and base $= B$.

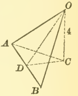
10. $\therefore$ vol. $EOD\text{–}ABC = HB$. Why?

11. $\therefore$ vol. $O\text{–}ABC = \frac{1}{3} HB$.

**Ex. 56.** If the base of a pyramid is a parallelogram, the plane determined by the vertex of the pyramid and a diagonal of the base divides the pyramid into two equal triangular pyramids.

**Ex. 57.** Determine the ratio to a given parallelopiped of the pyramid whose lateral edges are the three edges of the parallelopiped which intersect at any one corner.

**Ex. 58.** Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.

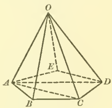



*Suggestion.* — In the figure, $C$ is the *center* of the base.

**Ex 59.** Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

## Proposition XVI. Theorem

**570.** *The volume of any pyramid is equal to one third the product of its base and altitude.*




The proof is like that for § 553.

Proof to be given by the student.

**571. Cor. 1.** *Any two pyramids having equal bases and equal altitudes are equal.*

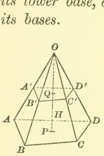
**Cor. 2.** *Two pyramids having equal altitudes have the same ratio as their bases.*

**Cor. 3.** *Two pyramids having equal bases have the same ratio as their altitudes.*

**Cor. 4.** *Any two pyramids have the same ratio as the products of their bases and altitudes.*

## Proposition XVII. Theorem

**572.** *The volume of a frustum of any pyramid is equal to one third of its altitude multiplied by the sum of its upper base, its lower base, and the mean proportional between its bases.*




**Hypothesis.** $V=$ the volume, $B=$ the area of the lower base, $b=$ the area of the upper base, and $H=$ the length of the altitude of $AC'$, a frustum of any pyramid $O\text{–}AC$.

**Conclusion.** $V=\frac{1}{3}H(B+b+\sqrt{Bb})$.

**Proof.** 1. Draw altitude $OP$, cutting $A'C'$ at $Q$.

2. $$V = \text{vol. } O\text{–}AC - \text{vol. } O\text{–}A'C'$$
$$= \tfrac{1}{3}B(H+OQ) - \tfrac{1}{3}b(OQ)$$
$$= \tfrac{1}{3}HB + \tfrac{1}{3}OQ(B-b).$$

3. But $B : b = \overline{OP}^2 : \overline{OQ}^2$. § 563

4. $\therefore \sqrt{B} : \sqrt{b} = OP : OQ$. Algebra

5. $\therefore (\sqrt{B}-\sqrt{b}) : \sqrt{b} = (OP - OQ) : OQ = H : OQ$. § 256

6. $\therefore OQ(\sqrt{B}-\sqrt{b}) = H\sqrt{b}$.

7. Multiplying both members by $\sqrt{B}+\sqrt{b}$,
$$\therefore OQ(B-b) = H(\sqrt{Bb}+b).$$

8. Substituting in step 2 for $OQ(B-b)$ its value from step 7,
$$V = \tfrac{1}{3}HB + \tfrac{1}{3}H(\sqrt{Bb}+b)$$
$$= \tfrac{1}{3}H(B+b+\sqrt{Bb}).$$

**Ex. 60.** Find the volume of a regular quadrangular pyramid each side of whose base is 3, and whose altitude is 5.

**Ex. 61.** Find the volume of a regular hexagonal pyramid each side of whose base is 4, and whose altitude is 9.

**Ex. 62.** The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

**Ex. 63.** Prove that the lines joining the center of a cube to the four vertices of one face are the edges of a regular quadrangular pyramid whose volume is $\frac{1}{6}$ that of the cube.

**Ex. 64.** Express the volume of a pyramid in terms of its altitude and the area of its mid-section parallel to the base.

**Ex. 65.** Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 18.

**Ex. 66.** Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

*Suggestion.* — Represent the lateral edge and the side of the base by $x$.

**Ex. 67.** Find the volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, respectively, and whose altitude is 10.

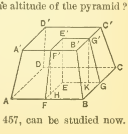
**Ex. 68.** Find the volume of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, respectively, and whose altitude is 24.

**Ex. 69.** Find the volume of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, respectively, and whose altitude is 12.

**Ex. 70.** A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height, each side of whose base is 2 ft. What is its weight, at 180 lb. to the cubic foot?

**Ex. 71.** The areas of the bases of a frustum of a pyramid are 12 and 75 respectively, and its altitude is 9. What is the altitude of the pyramid?

*Suggestion.* — Let the altitude of the pyramid $= x$; then $x - 9$ is the $\perp$ from its vertex to the upper base of the frustum; then use § 564.




**Ex. 72.** The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.

**Note.** — Supplementary Exercises 38–42, p. 457, can be studied now.

### SUPPLEMENTARY TOPICS

Five groups of supplementary material follow. None of this material is needed for subsequent parts of solid geometry. Each group is independent of each of the others. The groups are arranged in order of importance and of interest. The teacher should select from this material such parts as seem best to meet the needs of the class.

## Group A. Prismatoids

**573.** A **Prismatoid** is a polyedron bounded by two parallel faces called **Bases**, and by a number of lateral faces which are bounded by either triangles, trapezoids, or parallelograms.

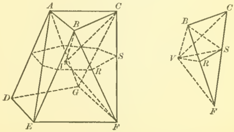
The **Altitude** of a prismatoid is the perpendicular between the bases.

The **Mid-section** of a prismatoid is the section of the plane parallel to the base and midway between them.

## Proposition XVIII. Theorem

**574.** *If the areas of the lower and upper bases of a prismatoid are $B$ and $b$, respectively, the area of the mid-section is $m$, the length of the altitude is $H$, and the volume is $V$, then,*

$$V = \frac{1}{6} H(B + b + 4m).$$




**Proof.** 1. Through any point $V$ of the mid-section and each edge of the prismatoid, pass planes. These planes divide the

prismatoid into pyramid $V\text{-}ABC$, pyramid $V\text{-}DEFG$, pyramids like $V\text{-}BCF$, and polyedra like $V\text{-}ABED$.

(*a*) Volume $V\text{-}ABC = \frac{1}{3} \cdot \frac{1}{2} H \cdot b = \frac{1}{6} Hb$.

(*b*) Similarly, volume $V\text{-}DEFG = \frac{1}{6} HB$.

(*c*) To compute the volume of $V\text{-}BCF$:

1. Draw $VR$ and $VS$, thus forming $\triangle VRS$, which is a part of the mid-section.

2. Draw plane $VBS$.

3. $V\text{-}BCF = V\text{-}RSF + V\text{-}BRS + V\text{-}BCS$.

4. $V\text{-}RSF = \frac{1}{3} \cdot \frac{1}{2} H \cdot \triangle VRS = \frac{1}{6} H \cdot \triangle VRS$. Prove it.

5. $V\text{-}BRS = \frac{1}{3} \cdot \frac{1}{2} H \cdot \triangle VRS = \frac{1}{6} H \cdot \triangle VRS$. Prove it.

6. $V\text{-}BCS = 2 \cdot V\text{-}BRS$, for $\triangle BCS = 2 \cdot \triangle BRS$. § 571, Cor. 2.
$\therefore V\text{-}BCS = \frac{2}{6} H \cdot \triangle VRS$.

7. $\therefore V\text{-}BCF = \frac{4}{6} H \cdot \triangle VRS$.

Similarly, the volume of any triangular pyramid with vertex $V$ and as base a triangular lateral face is equal to $\frac{4}{6} H$ multiplied by that part of $m$ which is in the triangular pyramid.

(*d*) $V\text{-}ABED$ can be divided into two triangular pyramids by passing a plane through $V$, $A$, and $E$. Hence its volume can be obtained as in part (*c*).

(*e*) Hence, the sum of all pyramids with vertex $V$, whose bases are lateral faces of the prismatoid, is $\frac{4}{6} H \cdot m$.

(*f*) $\therefore$ volume of the prismatoid $= \frac{1}{6} HB + \frac{1}{6} Hb + \frac{4}{6} Hm$
$= \frac{1}{6} H(B + b + 4m)$.

**Note.** — This Proposition is particularly interesting not alone because it enables us to determine the volumes of many irregularly shaped figures, but because it includes many previous propositions as special cases.

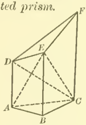
**Ex. 73.** Is a prism a special case of a prismatoid? In a prism, what relation is there between $B$, $b$, and $m$? Does the formula of § 574 reduce to the usual formula for the volume of a prism?

**Ex. 74.** Answer the same questions for a pyramid.

## Group B. Truncated Prisms

### Proposition XIX. Theorem

**575.** *A truncated triangular prism is equal to the sum of three pyramids having as common base the lower base of the given prism, and having as their vertices the three vertices of the upper base of the truncated prism.*



**Hypothesis.** $DEF–ABC$ is a truncated triangular prism.

**Conclusion.** $DEF–ABC = E–ABC + D–ABC + F–ABC.$

**Proof.** 1. Pass planes through $A$, $E$, and $C$, and through $D$, $E$, and $C$, thus dividing $DEF–ABC$ into $E–ABC$, $E–ADC$, and $E–DFC$.

2. $E–ABC$ is one of the required pyramids.

3. $E–DAC = B–DAC.$ Cor. 1, § 571

[For the altitudes from $B$ and $E$ to $ADC$ are equal.]

But $B–DAC \equiv D–ABC.$

$\therefore E–DAC = D–ABC$, the second required pyramid.

4. $E - DFC = B - AFC.$ § 571

[For $\triangle DFC = \triangle AFC$, and the altitudes from $E$ and $B$ to $DFC$ and $AFC$, respectively, are equal.]

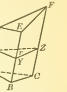
But $B–AFC \equiv F–ABC.$

$\therefore E–DFC = F–ABC$, the third required pyramid.

5. $\therefore DEF–ABC = E–ABC + D–ABC + F–ABC.$

**576. Cor. 1.** *The volume of a truncated right triangular prism is equal to one third the base multiplied by the sum of the lateral edges.*

**577. Cor.** 2. *The volume of any truncated triangular prism is equal to the product of one third the area of a right section by the sum of the lateral edges.*




*Suggestions.* — 1. Let $XYZ$ be the right section whose area is $r$.

2. Apply § 576, to $DEF\text{-}XYZ$, and to $ABC\text{-}XYZ$.

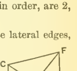
3. Prove $DEF\text{-}ABC = \frac{1}{3} r (AD + BE + CF)$.

**Ex. 75.** Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

**Ex. 76.** Find the volume of a truncated right triangular prism whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13, respectively.

**Ex. 77.** Find the volume of a truncated regular quadrangular prism, a side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

*Suggestion.* — Pass a plane through two diagonally opposite lateral edges, dividing the solid into two truncated right triangular prisms.

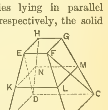



**Ex. 78.** If $ABCD$ is a rectangle, and $EF$ is any line not in its plane parallel to $AB$, the volume of the solid bounded by figures $ABCD$, $ABFE$, $CDEF$, $ADE$, and $BCF$, is

$$\tfrac{1}{6} h \times AD \times (2\,AB + EF),$$

where $h$ is the perpendicular from any point of $EF$ to $ABCD$. § 577

**Ex. 79.** If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, $AB$ and $BC$ being parallel to $EF$ and $FG$, respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid.*




$ABCD$ and $EFGH$ are called the bases of the rectangular prismoid, and the perpendicular distance between them, the altitude.

Prove the volume of a rectangular prismoid equal to the sum of its bases, plus four times a section equidistant from the bases, multiplied by one sixth the altitude.

*Suggestion.* — Pass a plane through $CD$ and $EF$, and find the volumes of the solids $ABCD\text{-}EF$ and $EFGH\text{-}CD$ by Ex. 78.

**Note.** — Supplementary Exercises 43–47, p. 458, can be studied now.

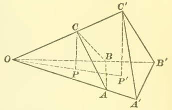
## Group C. Miscellaneous Theorems

The following theorems about tetraedra are very much like certain theorems about triangles and about triedral angles.

### Proposition XX. Theorem

**578.** *Two tetraedra having a triedral angle of one equal to a triedral angle of the other, have the same ratio as the products of the edges including the equal triedral angles.*

**Hypothesis.** $V$ and $V'$ are volumes of tetraedra $O\text{–}ABC$ and $O\text{–}A'B'C'$, respectively, having the common triedral $\angle O$.




**Conclusion.**

$$\frac{V}{V'}=\frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

**Proof.** 1. Draw lines $CP$ and $C'P' \perp$ to face $OA'B'$.

2. Let their plane intersect face $OA'B'$ in line $OPP'$.

3. Now, $OAB$ and $OA'B'$ are the bases, and $CP$ and $C'P'$ the altitudes, of triangular pyramids $C\text{–}OAB$ and $C'\text{–}OA'B'$, respectively.

4. $\therefore \dfrac{V}{V'}=\dfrac{\text{area } OAB \times CP}{\text{area } OA'B' \times C'P'}$ Why?

$=\dfrac{\text{area } OAB}{\text{area } OA'B'} \times \dfrac{CP}{C'P'}.$ (1)

5. But $\dfrac{\text{area } OAB}{\text{area } OA'B'}=\dfrac{OA \times OB}{OA' \times OB'}.$ § 346

6. Also $\triangle$ $OCP$ and $OC'P'$ are rt. $\triangle$. Why?

7. Then $\triangle$ $OCP$ and $OC'P'$ are similar. Why?

8. $\therefore \dfrac{CP}{C'P'}=\dfrac{OC}{OC'}.$ Why?

9. Substituting from steps 5 and 8 in step 4,

$$\frac{V}{V'}=\frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'}=\frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

**Ex. 80.** State the theorem of plane geometry about triangles which corresponds to Proposition XX.

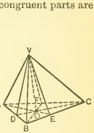
Note. — For each of the following exercises, state also the corresponding theorem about triangles.

**Ex. 81.** Prove that two tetraedra are congruent if a diedral angle and the adjacent faces of one are congruent, respectively, to a diedral angle and the adjacent faces of the other, if the congruent parts are arranged in the same order.

*Suggestion.* — Prove by superposition.

**Ex. 82.** Two tetraedra are congruent if three faces of one are congruent, respectively, to three faces of the other, if the congruent parts are arranged in the same order.

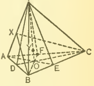
*Suggestion.* — Recall § 515.




**Ex. 83.** Prove that the three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.

**Ex. 84.** Prove that the six planes through the edges of a tetraedron bisecting the opposite edges meet in a common point.

*Suggestion.* — By Ex. 83, three planes meet in line $VO$. Let plane $XBC$ intersect $VO$ at $Y$. Prove $Y$ lies in the remaining two planes.




Note. — The common point is the *center of gravity* of the tetraedron.

**Ex. 85.** Prove that the center of gravity of a tetraedron divides the line drawn from any vertex to the center of gravity of the opposite face in the ratio 3 : 1.

**Ex. 86.** Prove that the six planes bisecting the diedral angles of a tetraedron meet in a common point.

Note. — Pupils will find it interesting to attempt to make up other theorems about tetraedra which are suggested by theorems about triangles.

## Group D. Regular Polyedra

**579.** A **Regular Polyedron** is a polyedron whose faces are congruent regular polygons, and whose polyedral angles are all congruent.

## Proposition XXI. Theorem

**580.** *Not more than five regular convex polyedra are possible.*

A convex polyedral ∠ must have at least three faces, and the sum of its face ∠s must be $< 360°$. § 514

1. *With equilateral triangles.*

Since each ∠ of an equilateral △ is 60°, we may form a convex polyedral ∠ by combining either 3, 4, or 5 equilateral △s.

Not more than 5 equilateral △s can be combined to form a convex polyedral ∠. § 514

Hence not more than three regular convex polyedra can be bounded by equilateral △s.

2. *With squares.*

Since each ∠ of a square is 90°, we may form a convex polyedral ∠ by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral ∠.

Hence not more than one regular convex polyedron can be bounded by squares.

3. *With regular pentagons.*

Since each ∠ of a regular pentagon is 108°, we may form a convex polyedral ∠ by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral ∠. Why?

Hence not more than one regular convex polyedron can be bounded by regular pentagons.

4. *With other regular polygons.*

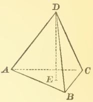
Since each ∠ of a regular hexagon is 120°, no convex polyedral ∠ can be formed by combining regular hexagons. Why?

Hence no regular convex polyedron can be bounded by regular hexagons.

In like manner, no regular convex polyedron can be bounded by regular polygons of more than six sides.

Therefore, not more than five regular convex polyedra are possible.

**Ex. 87.** Prove that the following construction produces a regular tetraedron having a given side.




**Construction.** — 1. With given side $AB$, construct equilateral $ABC$.

2. At its center $E$, draw $ED \perp ABC$.

3. Take $D$ on $DE$ so that $AD = AB$, and draw $AD$, $BD$, and $CD$.

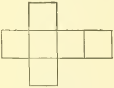
**Statement.** — $ABCD$ is a regular tetraedron.

Prove its faces are inclosed by congruent equilateral $\triangle$s and that its triedral angles are congruent.

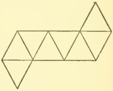
**581.** Models of the five regular polyedra may be made by drawing upon cardboard figures like the following.

Cut out the figures along the outer line; cut only halfway through the cardboard on the inner lines; bring the edges together and fasten them with gummed paper.

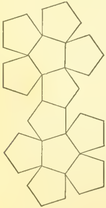
TETRAEDRON



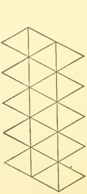
HEXAEDRON



OCTAEDRON



DODECAEDRON



ICOSAEDRON

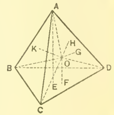
**Ex. 88.** Make a model of at least one of the regular polyedra.

**Ex. 89.** What is the sum of the face angles at any vertex of each of the regular polyedra ?

**Ex. 90.** Find the volume and the total area of a regular tetraedron whose edge is 10.

**Ex. 91.** Find the volume and the total area of a regular tetraedron whose edge is $a$.

**Ex. 92.** The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron is equal to its altitude.

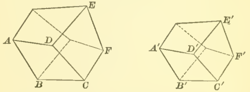



*Suggestion.* — Divide the tetraedron into triangular pyramids, having the given point for their common vertex. Find the volume of each and of the whole pyramid and form an equation.

**Ex. 93.** Prove that the volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

## Group E. Similar Polyedra

**582.** Two polyedra are **similar** when they have the same number of faces similar each to each and similarly placed, and have their homologous polyedral angles congruent.



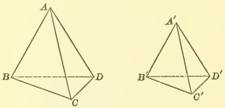

**Ex. 94.** Prove that the ratio of any two homologous edges of two similar polyedra is equal to the ratio of any other two homologous edges.

**Ex. 95.** Prove that any two homologous faces of two similar polyedra are to each other as the squares of any two homologous edges.

**Ex. 96.** Prove that the entire surfaces of two similar polyedra are to each other as the squares of any two homologous edges.

## Proposition XXII. Theorem

**583.** *Two tetraedra are similar when three face triangles including a triedral angle of one are similar, respectively, to three face triangles including a triedral angle of the other, and similarly placed.*




**Hypothesis.** In tetraedra $ABCD$ and $A'B'C'D'$

$\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, and $\triangle ADB \sim \triangle A'D'B'$.

**Conclusion.** $ABCD \sim A'B'C'D$.

**Proof.** 1. From the given similar $\triangle$s, we have

$$\frac{BC}{B'C'}=\left(\frac{AC}{A'C'}\right)=\frac{CD}{C'D'}=\left(\frac{AD}{A'D'}\right)=\frac{BD}{B'D'}.$$ Why?

2. Hence, $\triangle BCD \sim \triangle B'C'D'$. Why?

3. Again, $\angle$s $BAC$, $CAD$, and $DAB$ are equal, respectively, to $\angle$s $B'A'C'$, $C'A'D'$, and $D'A'B'$. Why?

4. Then, triedral $\angle$s $A\text{–}BCD$ and $A'\text{–}B'C'D'$ are congruent. § 516

5. Similarly, any two homologous triedral $\angle$s are congruent.

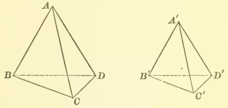
6. Therefore, $ABCD$ and $A'B'C'D'$ are similar. § 582

**Ex. 97.** Two tetraedra are similar when a diedral angle of one is congruent to a diedral angle of the other, and the face triangles including the congruent diedral angles are similar each to each, and similarly placed.

**Ex. 98.** If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.

## Proposition XXIII. Theorem

**584.** *Two similar tetraedra have the same ratio as the cubes of any two homologous edges.*



**Hypothesis.** $V$ and $V'$ are the volumes of similar tetraedra $ABCD$ and $A'B'C'D'$, respectively, $A$ and $A'$ being homologous vertices.

**Conclusion.** $$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

**Proof.** 1. Since $ABCD \sim A'B'C'D'$,

triedral $\angle A =$ triedral $\angle A'$.

2. $$\therefore \frac{V}{V'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}.$$ § 578

3. $$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}.$$

4. But $\frac{AC}{A'C'} = \frac{AB}{A'B'}$

and $\frac{AD}{A'D'} = \frac{AB}{A'B'}$. Why?

5. Substituting from step 4 in step 3,

$$\frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'}.$$

6. $$\therefore \frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

**Note** 1. It can be proved that any two similar polyedra can be divided into the same number of tetraedra, similar each to each and similarly placed. Consequently, it can be proved that any two similar polyedra have the same ratio as the cubes of any two homologous edges.

**Note** 2. It is interesting to notice that in similar figures

($a$) two homologous lines have the same ratio as any two homologous sides;

($b$) the areas of two homologous limited or bounded surfaces have the same ratio as the squares of any two homologous sides;

($c$) the volumes of two homologous solid parts have the same ratio as the cubes of any two homologous sides.

**Ex. 99.** The volume of a pyramid whose altitude is 7 in. is 686 cu. in. Find the volume of a similar pyramid whose altitude is 12 in.

**Ex. 100.** If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., find the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.

**Ex. 101.** Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

**Ex. 102.** A pyramid whose altitude is 10 in. weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

**Ex. 103.** An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface and the volume of the first polyedron.

**Ex. 104.** The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

*Suggestion.* — Let $x$ and $y$ denote the homologous edges of the tetraedra.

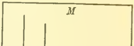
**Ex. 105.** The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

**Ex. 106.** The homologous edges of three similar tetraedra are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

*Suggestion.* — Represent the edge by $x$.

## BOOK VIII

### THE CYLINDER AND THE CONE

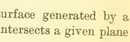



**585. Generating a Surface.** If a straight line $l$ moves so that it constantly intersects a straight line $b$ and is constantly parallel to a straight line $a$ which intersects $b$, it can be proved that $l$ constantly lies in the plane $M$ determined by $a$ and $b$; also it can be proved that every point in $M$ lies in line $l$ at some time during its period of movement.

Line $l$ is said to *generate the plane M*.

By suitable movement of a straight line, the straight line can be made to generate various curved surfaces as well as a plane surface.

**586.** A **Cylindrical Surface** is the surface generated by a moving straight line which constantly intersects a given plane curve and which is constantly parallel to another given straight line not in the plane of the curve.



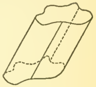

Thus, if $AB$ moves so as constantly to intersect plane curve $AD$ and to be constantly parallel to $MN$, not in the plane of $AD$, then $AB$ generates the cylindrical surface $BD$.

The moving line is called the **Generatrix**; the curved line is called the **Directrix**. Any position of the generatrix, as $EF$, is called an **Element** of the cylindrical surface.

**Ex. 1.** How many elements does a cylindrical surface have?

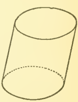
**Ex. 2.** Consider any two elements of a cylindrical surface. What kind of lines are they?

**Ex. 3.** Can more than one cylindrical surface be generated by use of a given directrix?



**587.** If the directrix is a *closed* plane curve, the cylindrical surface separates an infinite part of space from surrounding space.

The cylindrical surfaces considered in this text always have closed convex directrices.



**588.** A **Cylinder** is the solid (§ 525) bounded by a portion of a cylindrical surface and by portions of two parallel planes which intersect all the elements of the surface.

The portions of the parallel planes are the **Bases** of the cylinder; and the portion of the cylindrical surface between the planes is the **Lateral Surface** of the cylinder.

The **Total Surface** of a cylinder consists of its lateral surface and its bases.

The perpendicular between the two parallel planes is the **Altitude** of the cylinder. The segment of an element of the cylindrical surface which lies between the bases is an **Element** of the cylinder.

A **Section** of a cylinder is the part of the cylinder common to it and a plane cutting all the elements of the cylinder.

If a section of a cylinder is made by a plane perpendicular to the elements, it is a **Right Section** of the cylinder.

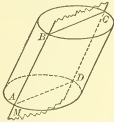
**Ex. 4.** The elements of a cylinder are equal and parallel.

**589. Kinds of Cylinders.** A **Right Cylinder** is a cylinder whose elements are perpendicular to its bases.

A **Circular Cylinder** is a cylinder whose base is inclosed by a circle.

## Proposition I. Theorem

**590.** *If a plane passes through an element of a cylinder and through at least one other point of the surface of the cylinder, the intersection with the total surface of the cylinder is a parallelogram.*




**Hypothesis.** Plane $M$, passing through element $AB$ of cylinder $AF$, intersects the lateral surface of $AF$ in point $C$, not in $AB$.

**Conclusion.** The section of $AF$ made by plane $M$ is inclosed by a parallelogram.

**Proof.** 1. Through $C$ draw $CD \parallel AB$.

2. $\therefore$ $CD$ is an element of surface $AF$, intersecting the bases at $C$ and $D$ respectively. § 586

3. $CD$ is also in $M$. § 447, I and III

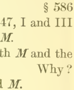
4. $\therefore$ $CD$ is in the intersection of $AF$ and $M$.

5. $BC$ and $AD$ are $\parallel$ straight lines, lying in both $M$ and the bases of $AF$. Why?

6. $\therefore$ $ABCD$ is the intersection of $AF$ and $M$.

7. Also $ABCD$ is a $\square$. Why?

**Ex. 5.** If a rectangle revolves about one of its sides as an axis, it generates a right circular cylinder.

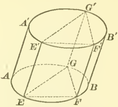



*Suggestion.* — Prove that $C$ describes a circle, that $BC$ generates a cylinder, and that the cylinder is a right cylinder.

**591.** As a consequence of Ex. 5, a right circular cylinder is also called a *cylinder of revolution.*

## Proposition II. Theorem

**592.** *The bases of a cylinder are congruent.*




**Hypothesis.** $AB'$ is any cylinder, with bases $A'B'$ and $AB$.

**Conclusion.** Base $A'B' \cong$ base $AB$.

**Proof.** 1. Let $E'$ and $F'$ be two particular points of curve $A'B'$ and $G'$ any other point. Draw elements $EE'$, $FF'$, and $GG'$; also draw $EF$, $FG$, $EG$, $E'F'$, $F'G'$, and $E'G'$.

2. $\therefore \triangle EFG \cong \triangle E'F'G'$. Prove it.

3. $\therefore$ base $A'B'$ may be superposed on base $AB$ so that $E'$, $F'$, and $G'$ will fall upon $E$, $F$, and $G$ respectively.

4. But $G'$ was any point of $A'B'$ except $E'$ and $F'$.

5. $\therefore E'$, $F'$, and every other point of curve $A'B'$ will fall upon a corresponding point of curve $AB$.

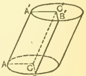
6. $\therefore$ base $A'B' \cong$ base $AB$. Why?

**Note.** — It is obvious that every point of curve $AB$ can be proved to fall upon a corresponding point of $A'B'$.

**Ex. 6.** Prove that a section of a circular cylinder made by a plane parallel to the base is inclosed by a circle.

**Note.** — A section of a circular cylinder made by a plane not parallel to the base is inclosed by a curve called an *ellipse*.

**Ex. 7.** Prove that a line drawn parallel to the elements of a circular cylinder from the center of one base intersects the other base at its center.




*Suggestions.* — 1. Let $A'$ be a particular point and $B'$ be any other point of the bounding circle of the upper base whose center is $C'$.

2. Draw $C'C \parallel B'B$, and draw element $A'A$.

3. Prove that $CB = CA$, and that $C$ is the center of curve $AB$.

**593.** The **Axis** of a circular cylinder is a straight line drawn between the centers of its bases.

**Ex. 8.** Prove that the axis of a circular cylinder is parallel to the elements of the lateral surface.

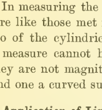
**Ex. 9.** Prove that the axis of a circular cylinder passes through the centers of all sections parallel to the bases.

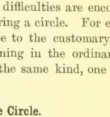
### MEASURING THE CYLINDER

**594.** In measuring the cylinder, difficulties are encountered which are like those met in measuring a circle. For example, the ratio of the cylindrical surface to the customary unit of surface measure cannot have meaning in the ordinary sense since they are not magnitudes of the same kind, one being a plane and one a curved surface.

**595. Application of Limits to the Circle.**

(*a*) The circumference of a circle, *i.e.* the length of a circle, is defined to be the limit of the perimeter of any regular inscribed polygon as the number of sides is indefinitely increased. § 412




(*b*) It is proved that the perimeter of any regular circumscribed polygon (as well as inscribed polygon) approaches the circumference of the circle as limit if the number of sides is indefinitely increased. § 413

(*c*) It can be proved that the perimeter of *any* inscribed or circumscribed polygon approaches the circumference of the circle as limit if the number of sides is increased indefinitely in such manner that the length of every side approaches the limit zero.

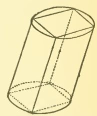
(*d*) The area of a circle is defined to be the limit of the area of any regular inscribed polygon as the number of sides increases indefinitely. § 417

(*e*) It is proved that the area of any regular circumscribed polygon approaches the area of the circle as limit as the number of sides is increased indefinitely. § 418

(*f*) It can be proved that the area of *any* inscribed or circumscribed polygon approaches the area of the circle as limit if the number of sides is increased indefinitely in such manner that the length of each side approaches the limit zero.

**Note.** — Remember that $\doteq$ is the symbol for "approaches the limit."

**596.** A *prism* is *inscribed in a cylinder* when its lateral edges are elements of the cylinder and its bases are in the planes of the bases of the cylinder. The polygons bounding the bases of the prism are inscribed in the boundaries of the bases of the cylinder.



**597. Application of Limits to a Cylinder.** Inscribe in a circular cylinder a prism having its base inclosed by a regular polygon; then inscribe a second prism whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely. Pass a plane forming right sections of the cylinder and the prisms.

It will be assumed evident that the prisms come nearer and nearer to occupying the same space as the cylinder. As a consequence:

(*a*) The **Volume of the Cylinder** is defined to be the limit of the volume of the inscribed prism as the number of faces increases indefinitely.

(*b*) The **Lateral Area of a Cylinder** is defined to be the limit of the lateral area of the inscribed prism as the number of faces increases indefinitely.

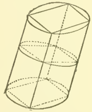
(*c*) The length of a right section of the lateral surface of the cylinder is defined to be the limit of the length of the right

section of the lateral surface of the inscribed prism as the number of faces increases indefinitely.

(*d*) It is evident that the edge and altitude of the inscribed prism equal respectively the element and altitude of the cylinder.

## Proposition III. Theorem

**598.** *The lateral area of a circular cylinder is equal to the perimeter of a right section multiplied by the length of an element.*



**Hypothesis.** $S =$ the lateral area, $P =$ the perimeter of a right section, $E =$ the length of an element of a circular cylinder.

**Conclusion.** $S = E \times P.$

**Proof.** 1. Inscribe in the cylinder a prism whose base is inclosed by a regular polygon.

Let $S' =$ the lateral area and $P' =$ the perimeter of a right section.

2. Then $S' = E \times P'.$ Why?

3. Let the number of faces of the prism increase indefinitely. Then

$S' \doteq S$, and $P' \doteq P.$ § 597, *b* and *c*; also Note, § 595.

4. $\therefore E \times P' \doteq E \times P.$ § 543, *a*

5. $\therefore S = E \times P.$ § 543, *b*

**599. Cor. 1.** *The lateral area of a right circular cylinder is equal to the circumference of its base multiplied by the length of the altitude.*

**600. Cor.** 2. *If $R$ = the length of the radius of the base, $H$ = the length of the altitude, $S$ = the lateral area, and $T$ = the total area of a right circular cylinder, then*

(*a*) $S = 2\pi RH.$

(*b*) $T = 2\pi R^2 + 2\pi RH = 2\pi R(R + H).$

**Ex. 10.** Find the lateral area of a right circular cyliuder whose altitude is 16 and the diameter of whose base is 18.

**Ex. 11.** Find the total area of a cylinder of revolution whose altitude is 15 and the radius of whose base is 5.

**Ex. 12.** Determine the cost at 15¢ per square yard of painting the vertical surface and top of a gas holder whose diameter is 30 ft. and whose height is 20 ft.

**Ex. 13.** How many square feet of tin are required to make 30 sections of hot-air furnace pipe 10 in. in diameter and 30 in. in length?

## Proposition IV. Theorem

**601.** *The volume of a circular cylinder is equal to the area of its base multiplied by the length of its altitude.*

**Hypothesis.** $V$ = the volume, $B$ = the area of the base, and $H$ = the length of the altitude of a circular cylinder.

**Conclusion.** $V = H \times B.$

**Proof.** 1. Inscribe in the cylinder a prism having its base inclosed by a regular polygon. Let $V'$ = the volume and $B'$ = the area of the base.

2. $\therefore V' = H \times B'.$ Why?

3. Increase indefinitely the number of faces of the prism. Then $V' \doteq V$, and $B' \doteq B.$

4. $\therefore H \times B' \doteq H \times B.$ § 543, *a*.

5. $\therefore V = H \times B.$ § 543, *b*.

**602. Cor.** *If $V$ = the volume, $H$ = the length of the altitude, and $R$ = the radius of the base of a circular cylinder, then*

$$V = \pi R^2 H.$$

**Ex. 14.** What is the cost of digging a dry well 5 ft. in diameter and 15 ft. deep at 50 ¢ per cubic yard?

**Ex. 15.** What is the capacity in gallons of a water tank 12 ft. in length and 36 in. in diameter, estimating $7\frac{1}{2}$ gal. of water to a cubic foot?

**Ex. 16.** How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in. and whose thickness is 1 in.?

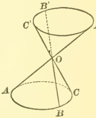
**Ex. 17.** Determine the number of cubic yards of concrete required for the wall and floor of a circular cistern 8 ft. in outside diameter, and 12 ft. deep, if the walls and floor are 8 in. thick.

**Ex. 18.** Determine the diameter of a 2-bbl. water reservoir having the form of a right circular cylinder if the length is 4 ft. (2 bbl. = 63 gal.; 1 cu. ft. contains $7\frac{1}{2}$ gal.)

**Note.** — Supplementary Exercises 48–58, p. 459, can be studied now.

## THE CONE

**603.** A **Conical Surface** is the surface generated by a moving straight line, which constantly intersects a given plane curve and constantly passes through a given point not in the plane of the curve.




Thus, if line $OA$ moves so as constantly to intersect plane curve $ABC$, and constantly passes through point $O$, not in the plane of the curve, it generates a conical surface.

The moving line is called the **Generatrix**, and the curve the **Directrix**.

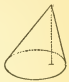
The given point is called the **Vertex**, and any position of the generatrix, as $OB$, is called an **Element** of the surface.

If the generatrix be supposed to be indefinite in length, it will generate two conical surfaces of indefinite extent, $O$–$A'B'C'$ and $O$–$ABC$.

These are called the *upper* and *lower nappes*, respectively, of the conical surface.

It will be assumed that the directrix is a closed plane curve so that each nappe of the surface separates an infinite portion of space from surrounding space.

**604.** A **Cone** is a solid bounded by a portion of one nappe of a conical surface and that part of a plane cutting all the elements of the surface which lies within the surface.



The plane is called the **Base** of the cone, and the conical surface the **Lateral Surface.**

The **Altitude** of a cone is the perpendicular from the vertex to the plane of the base.

**605. Kinds of Cones.**

A **Circular Cone** is a cone whose base is inclosed by a circle.

The **Axis** of a circular cone is a straight line drawn from the vertex to the center of the base.

A **Right Circular Cone** is a circular cone whose axis is perpendicular to its base.

A **Frustum of a Cone** is a portion of a cone included between the base and a plane parallel to the base.

The base of the cone is called the *lower base,* and the section made by the plane the *upper base, of the frustum.*

The *altitude of a frustum* is the perpendicular between the planes of the bases.

**Ex. 19.** Prove that the elements of a right circular cone are equal.

**Ex. 20.** Prove that the elements of a frustum of a right circular cone are equal.

**Ex. 21.** If a right triangle be revolved about one of its legs as an axis, it generates a right circular cone.

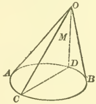
**606**. As a consequence of Ex. 19, the distance from the vertex to any point of the circle bounding the base of a right circular cone is called the **Slant Height** of the right circular cone.

As a consequence of Ex. 20, the portion of the slant height of a right circular cone between the base and a plane parallel to the base is called the **Slant Height** of the frustrum of the right circular cone.

As a consequence of Ex. 21, a right circular cone is also called a cone of revolution.

## Proposition V. Theorem

**607.** *If a plane passes through an element of a cone and through at least one other point of the surface of the cone, the intersection with the total surface of the cone is a triangle.*




**Hypothesis.** Plane $M$, passing through element $OC$ of cone $O\text{–}AB$, intersects the surface again in point $D$, not in $OC$.

**Conclusion.** Section $OCD$ is a $\triangle$.

**Proof.** 1. $CD$ is a straight line. Why?

2. Since the base $AB$ is inclosed by a closed line, line $CD$ intersects it in two points $C$ and $D$ which lie in the conical surface.

3. Lines $OC$ and $OD$ are elements of the surface. Def.

4. Also $OC$ and $OD$ lie in the plane $OCD$. Why?

5. $\therefore$ the complete intersection of the surface and the plane is the triangle $OCD$.

**Ex. 22.** Find, correct to one decimal, the slant height of a right circular cone whose altitude is 9 in. and the radius of whose base is 3 in.

**Ex. 23.** Find the slant height of a right circular cone whose altitude is $h$ and the radius of whose base is $r$.

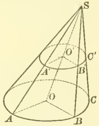
**Ex. 24.** The radii of the upper and lower bases of the frustum of a right circular cone are 3 in. and 5 in. respectively; the altitude of the frustum is 6 in. Determine the slant height correct to one decimal place.

**Ex. 25.** Find the altitude of a right circular cone whose slant height is 13 and the radius of whose base is 5.

**Ex. 26.** Repeat Ex. 24 when the radii are $r$ and $R$ respectively and the altitude is $H$.

## Proposition VI. Theorem

**608.** *The section of the lateral surface of a circular cone made by a plane parallel to the base is a circle.*




**Hypothesis.** $A'B'C'$ is the section of circular cone $S$–$ABC$, made by a plane ∥ base $ABC$ whose center is $O$.

**Conclusion.** $A'B'C'$ is a ⊙.

**Proof.** 1. Draw axis $OS$, intersecting $A'B'C'$ at $O'$.

2. Let $A'$ and $B'$ be any two points in curve $A'B'C'$. Let planes $A'O'S$ and $B'O'S$ intersect the base in radii $OA$ and $OB$, the cutting plane in lines $O'A'$ and $O'B'$, and the lateral surface in lines $SA'A$ and $SB'B$ respectively.

3. $SA'A$ and $SB'B$ are straight lines. § 607

4. $\triangle SA'O' \sim \triangle SAO$, and $\triangle SB'O' \sim \triangle SBO$. Prove it.

5. $\therefore \dfrac{O'A'}{OA} = \dfrac{O'B'}{OB}$. Prove it.

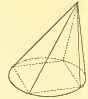
6. $\therefore O'A' = O'B'$. Prove it.

7. Since $A'$ and $B'$ are any two points of curve $A'B'C'$ and are equidistant from $O'$, curve $A'B'C'$ is a circle and $O'$ is its center.

**609. Cor.** *The axis of a circular cone passes through the center of every section parallel to the base.*

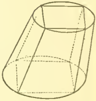
**Ex. 27.** Prove that the radii of the upper and lower bases of a frustum of any cone have the same ratio as the distances of the bases from the vertex of the cone.

## MEASURING THE CONE



**610.** A **pyramid** is **inscribed in a cone** when its lateral edges are elements of the cone and the bounding polygon of the base of the pyramid is inscribed in the boundary of the base of the cone. The vertex and altitude of the pyramid coincide with the vertex and the altitude of the cone.

**611.** A **frustum of a pyramid** is **inscribed in a frustum of a cone** when its lateral edges are elements of the frustum of the cone, and the boundaries of the bases of the frustum of the pyramid are inscribed in the boundaries of the bases of the frustum of the cone. The altitude of the frustum of the pyramid coincides with the altitude of the frustum of the cone.



**612. Application of Limits to the Cone.** Inscribe in a circular cone a pyramid having a base inclosed by a regular polygon; then inscribe a second pyramid whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely.

It will be assumed evident that the pyramids come nearer and nearer to occupying the same space as the cone. As a consequence:

(*a*) The **Volume of a Cone** is defined to be the limit of the volume of a regular inscribed pyramid as the number of faces is increased indefinitely.

(*b*) The **Lateral Area of a Cone** is defined to be the limit of the lateral area of a regular inscribed pyramid as the number of faces is increased indefinitely.

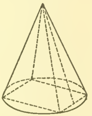
(*c*) It can be proved that the slant height of a right circular cone is the limit of the slant height of a regular inscribed pyramid as the number of faces is increased indefinitely.

(*d*) The volume and lateral area of a frustum of a cone are

defined to be the limits of the volume and area respectively of the frustum of a regular inscribed pyramid, having its base inclosed by a regular polygon, as the number of faces is increased indefinitely; also, the slant height of a frustum of a right circular cone can be proved to be the limit of the slant height of the frustum of a regular inscribed pyramid as the number of faces is increased indefinitely.

## Proposition VII. Theorem

**613**. *The lateral area of a right circular cone is equal to the circumference of its base multiplied by one half its slant height.*



**Hypothesis.** $S =$ the lateral area, $C =$ the circumference of the base, and $L =$ the slant height of a right circular cone.

**Conclusion.** $S = \frac{1}{2} CL$.

**Proof.** 1. Inscribe in the cone a pyramid whose base is inclosed by a regular polygon. Let $S' =$ the lateral area, $P' =$ the perimeter of the base, and $L' =$ the slant height of the pyramid.

2. Then $S' = \frac{1}{2} P'L'$. Why?

3. Let the number of faces of the pyramid increase indefinitely, keeping the pyramid always a regular pyramid. Then

$S' \doteq S,\ P' \doteq C$, and $L' \doteq L$. § 612, *b, c*

4. $\therefore \frac{1}{2} P'L' \doteq \frac{1}{2} CL$. § 543, *c*

5. $\therefore S = \frac{1}{2} CL$ § 542, *b*

**614. Cor.** *If $S$ denotes the lateral area, $T$ the total area, $L$ the slant height, and $R$ the radius of the base of a right circular cone,* then

$$S = 2\pi R \times \tfrac{1}{2} L = \pi RL.$$

Also, $$T = \pi RL + \pi R^2 = \pi R(L + R).$$

**Ex. 28.** Find the lateral area and the total area of a right circular cone, the radius of whose base is 7 in. and whose slant height is 25 in.

**Ex. 29.** How many square yards of canvas are required for a circus tent having the form of a right circular cylinder, surmounted by a right circular cone, if the diameter of the tent is 100 ft., the height of the vertical wall 15 ft., and the height of the highest point of the tent 50 ft.?

**Ex. 30.** The diameter of the base of a right circular cone is equal to its altitude. Determine its lateral and total area.

## Proposition VIII. Theorem

**615.** *The volume of a circular cone is equal to the area of its base multiplied by one third the length of its altitude.*

**Hypothesis.** $V =$ the volume, $B =$ the area of the base, and $H =$ the altitude of a circular cone.

**Conclusion.** $V = \frac{1}{3} BH.$

Proof left to the pupil.

*Suggestion.* — Model the proof after that in § 601. Use Fig. § 613

**616. Cor.** *If $V$ denotes the volume, $H$ the altitude, and $R$ the radius of the base of a circular cone,*

$$V = \tfrac{1}{3}\pi R^2 H.$$

**Ex. 31.** Determine the volume of a right circular cone, the radius of whose base is 7 in. and whose slant height is 25 in.

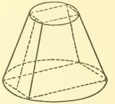
**Ex. 32.** Determine the volume of the solid generated when a right triangle of base $b$ and altitude $h$

($a$) revolves about its side $h$; ($b$) revolves about its side $b$.

**Ex. 33.** Determine the ratio of a circular cone and a circular cylinder having the same base and altitude.

## Proposition IX. Theorem

**617.** *The lateral area of a frustum of a right circular cone is equal to the sum of the circumferences of its bases, multiplied by one half its slant height.*



**Hypothesis.** $S =$ the lateral area, $C$ and $c$ the circumferences of the lower and upper bases respectively, and $L =$ the slant height of a frustum of a right circular cone.

**Conclusion.** $S = \frac{1}{2} L(C + c)$.

**Proof.** 1. Let $S' =$ the lateral area, $C'$ and $c'$ the circumferences of the lower and upper bases, and $L' =$ the slant height of the frustum of a regular pyramid inscribed in the frustum of the cone.

2. $\therefore S' = \frac{1}{2} L'(C' + c')$. Why?

3. Let the number of faces of the frustum of the pyramid be increased indefinitely, keeping the pyramid always a regular pyramid. Then

$S' \doteq S,\ C' \doteq C,\ c' \doteq c,\ L' \doteq L.$ Why?

4. $\therefore \frac{1}{2} L'(C' + c') \doteq \frac{1}{2} L(C + c)$. § 543, *c*

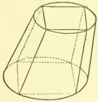
5. $\therefore S = \frac{1}{2} L(C + c)$. § 543, *b*

**618. Cor. 1.** *If $S$ denotes the lateral area, $L$ the slant height, and $R$ and $r$ the radii of the bases of a frustum of a right circular cone,* $S = (2\pi R + 2\pi r) \times \frac{1}{2} L = \pi(R + r)L$.

**619. Cor. 2.** *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases, multiplied by the slant height.*

## Proposition X. Theorem

**620.** *The volume of a frustum of a circular cone is equal to the sum of its bases and the mean proportional between them, multiplied by one third the length of its altitude.*



**Hypothesis.** $V =$ the volume, $B$ and $b =$ the areas of the lower and upper bases respectively, and $H =$ the length of the altitude of a frustum of a circular cone.

**Conclusion.** $V = \frac{1}{3}H(B + b + \sqrt{Bb})$.

**Proof.** 1. Inscribe in the frustum of the cone a frustum of a pyramid having its base inclosed by a regular polygon. Let $V' =$ the volume, and $B'$ and $b' =$ the areas of the lower and upper bases respectively of the frustum of the pyramid.

2. Then $V' = \frac{1}{3}H(B' + b' + \sqrt{B'b'})$. § 572

Complete the proof as in § 601 and § 615.

**621. Cor.** *If $V$ denotes the volume, $H$ the altitude, and $R$ and $r$ the radii of the bases of a frustum of a circular cone, then*

$$V = \tfrac{1}{3}\pi(R^2 + r^2 + Rr)H.$$

**Ex. 34.** Find the lateral area, the total area, and the volume of a frustum of a cone of revolution, the diameters of whose bases are 16 in. and 6 in., and whose altitude is 12 in.

**Ex. 35.** Determine the contents in quarts of a water pail having the form of a frustum of a cone of revolution, if the diameters of the bottom and top are 9 in. and 12 in. respectively, and the height of the pail is 14 in. (One quart occupies about 231 cu. in.)

**Note.** — Supplementary Exercises 59–71, p. 459, can be studied now.

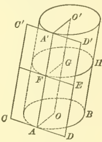
SUPPLEMENTARY TOPICS

## A. Planes Tangent to a Cylinder or a Cone

**622.** A **plane** is **tangent** to a circular cylinder or to a circular cone when it contains one and only one element of the cylinder or of the cone.

## Proposition XI. Theorem

**623.** *A plane drawn through an element of a circular cylinder and a tangent to the base at its extremity is tangent to the cylinder.*




**Hypothesis.** $AA'$ is an element of the lateral surface of circular cylinder $AB'$, line $CD$ is tangent to the base $AB$ at $A$, and plane $CD'$ is drawn through $AA'$ and $CD$.

**Conclusion.** $CD'$ is tangent to the cylinder.

**Proof.** 1. Let $E$ be any point in plane $CD'$, not in $AA'$, and draw through $E$ a plane $\parallel$ to the bases, intersecting $CD'$ in line $EF$ and the cylinder in $FH$.

2. Draw axis $OO'$; then $OO'$ is $\parallel AA'$. Ex. 8, p. 392.

3. Let the plane of $OO'$ and $AA'$ intersect the planes of $AB$ and $FH$ in radii $OA$ and $GF$, respectively.

4. Then, $GF \parallel OA$ and $FE \parallel AD$. Why?

5. $\therefore \angle GFE = \angle OAD$. Why?

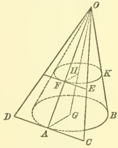
6. $\therefore FE \perp GF$, and tangent to $\odot FH$. Prove it.

7. Whence, point $E$ lies outside the cylinder.

8. $\therefore$ all points of $CD'$, not in $AA'$, lie outside the cylinder, and $CD'$ is tangent to the cylinder.

## Proposition XII. Theorem

**624.** *A plane determined by an element of the lateral surface of a circular cone and a tangent to the base at its extremity, is tangent to the cone.*




**Hypothesis.** $OA$ is an element of the lateral surface of circular cone $OAB$, line $CD$ is tangent to base $AB$ at $A$, and plane $OCD$ is drawn through $OA$ and $CD$.

**Conclusion.** $OCD$ is tangent to the cone.

(Prove that $E$ lies outside the cone.)

*Suggestion.* — Model the proof after that of § 623.

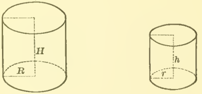
## B. Similar Cylinders and Cones of Revolution

**625. Similar cylinders of revolution** are right circular cylinders generated by the revolution of similar rectangles about homologous sides as axes.

**Similar cones of revolution** are right circular cones generated by the revolution of similar right triangles about homologous sides as axes.

## Proposition XIII. Theorem

**626.** *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*




**Hypothesis.** $S$ and $s$ are the lateral areas, $T$ and $t$ are the total areas, $V$ and $v$ are the volumes, $H$ and $h$ are the altitudes, and $R$ and $r$ the radii of the bases, of two similar cylinders of revolution.

**Conclusion.** $\dfrac{S}{s}=\dfrac{T}{t}=\dfrac{H^2}{h^2}=\dfrac{R^2}{r^2}$, and $\dfrac{V}{v}=\dfrac{H^3}{h^3}=\dfrac{R^3}{r^3}$.

**Proof.** 1. Since the generating rectangles are similar,

$$\frac{H}{h}=\frac{R}{r}. \qquad \text{Why?}$$

2. $$\therefore \frac{H}{h}=\frac{R}{r}=\frac{H+R}{h+r}. \qquad \S\,255,\ \S\,253$$

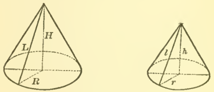
3. $$\frac{S}{s}=\frac{2\pi RH}{2\pi rh}=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{H^2}{h^2}. \qquad \text{Why?}$$

4. $$\frac{T}{t}=\frac{2\pi R(H+R)}{2\pi r(h+r)}=\frac{R}{r}\times\frac{R}{r}=\frac{R^2}{r^2}=\frac{H^2}{h^2}. \qquad \text{Why?}$$

5. $$\frac{V}{v}=\frac{\pi R^2H}{\pi r^2h}=\frac{R^2}{r^2}\times\frac{R}{r}=\frac{R^3}{r^3}=\frac{H^3}{h^3}. \qquad \text{Why?}$$

## Proposition XIV. Theorem

**627.** *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*




**Hypothesis.** $S$ and $s$ are the lateral areas, $T$ and $t$ the total areas, $V$ and $v$ are the volumes, $L$ and $l$ are the slant heights, $H$ and $h$ are the altitudes, and $R$ and $r$ the radii of the bases, of two similar cones of revolution. (§ 625.)

**Conclusion.** $\frac{S}{s}=\frac{T}{t}=\frac{L^2}{l^2}=\frac{H^2}{h^2}=\frac{R^2}{r^2}$, and $\frac{V}{v}=\frac{L^3}{l^3}=\frac{H^3}{h^3}=\frac{R^3}{r^3}$.

The proof is left to the pupil; model it after that of § 626.

**Ex. 36.** At what distance from the vertex of a right circular cone with altitude $H$ must a plane parallel to the base be passed so that the lateral area will be bisected?

## BOOK IX

### THE SPHERE

**628.** A **Spherical Surface** is a closed surface all points of which are equidistant from a point within, called the **Center**.

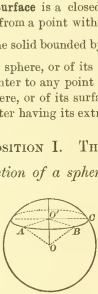
**629.** A **Sphere** is the solid bounded by a spherical surface.

**630.** A **Radius** of a sphere, or of its surface, is the straight line drawn from its center to any point of its surface.

A **Diameter** of a sphere, or of its surface, is the straight line drawn through the center having its extremities in the surface.

### Proposition I. Theorem

**631.** *The intersection of a spherical surface and a plane is a circle.*




**Hypothesis.** $ABC$ is the intersection of the spherical surface whose center is $O$ and a plane.

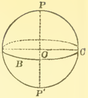
**Conclusion.** Curve $ABC$ is a $\odot$

**Proof.** 1. Draw $OO' \perp$ plane of $ABC$.

2. Let $A$ and $B$ be any two points in curve $ABC$. Draw $OA$, $OB$, $O'A$, and $O'B$.

*Suggestion.* — Now prove that $O'A = O'B$, and $ABC$ is a $\odot$.

**632.** A **Great Circle** of a sphere is the intersection of its surface and a plane passing through its center; as $\odot ABC$.




A **Small Circle** of a sphere is the intersection of its surface and a plane which does not pass through its center.

The **Axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle; as axis $POP'$.

The **Poles of a Circle** of a sphere are the extremities of the axis of the circle.

**633. Cor. 1.** *The axis of a circle of a sphere passes through the center of the circle.*

**634. Cor. 2.** *All great circles of a sphere are equal.*

**635. Cor. 3.** *Every great circle bisects the sphere and its surface.*

For if the portions of the sphere formed by the plane of the great circle be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either surface are equally distant from the center.

**636. Cor. 4.** *Any two great circles bisect each other.*

For the intersection of their planes is a diameter of the sphere, and therefore a diameter of each circle.

**637. Cor. 5.** *Between any two points on the surface of a sphere, not the extremities of a diameter, one and only one arc of a great circle, less than a semicircle, can be drawn.*

For the two points, with the center of the sphere, determine a plane which intersects the surface of the sphere in the required arc.

**638.** *Two spheres are equal when their radii are equal.*

*All radii and diameters of the same sphere or equal spheres are equal.*

**639.** A spherical surface may be generated by the revolution of a semicircle about its diameter as an axis.

For all points of such a surface are equally distant from the center of the circle.

**640.** Through two points of a spherical surface, an infinity of small circles of the sphere can be drawn. Why?

Through three points of a spherical surface, one and only one small circle of the sphere can be drawn. Why?

**Ex. 1.** Prove that a great circle of a sphere which passes through one pole of a circle must pass through the other pole also.

**Ex. 2.** How many great circles can be passed through two points which are the extremities of a diameter of a sphere?

**Ex. 3.** What kind of circles of the earth are the parallels of latitude?

**Ex. 4.** What kind of circles of the earth are the meridians?

**Ex. 5.** What kind of circle of the earth is the equator?

**Ex. 6.** Speaking strictly, is it accurate to speak of the North Pole of the earth? Of what circle or circles is it a pole?

**Ex. 7.** Into how many parts do two great circles of a sphere divide the surface of the sphere?

**Ex. 8.** Into how many parts do three great circles of a sphere divide the surface of the sphere, if they do not all have a common diameter?

**Ex. 9.** Prove that all circles of a sphere made by parallel planes have the same axis and the same poles.

**Ex. 10.** Given a point of a spherical surface. Prove that it is the pole of one and only one great circle.

**Ex. 11.** Prove that circles of a sphere made by planes equidistant from the center of the sphere are equal.

*Suggestion.* — Use the Pythagorean Theorem. (§ 291.)

**Ex. 12.** State and prove the converse of Ex. 11.

**Ex. 13.** Prove that circles of a sphere made by planes unequally distant from the center of the sphere are unequal, the more remote being the smaller.

**Ex. 14.** State and prove the converse of Ex. 13.

**Ex. 15.** In how many points can two straight lines intersect?

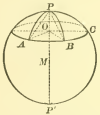
In how many points on one hemisphere can two great circles intersect?

**641.** The great circles of a sphere in *spherical geometry* correspond to the straight lines of a plane in *plane* geometry. § 637 and Ex. 15 are two instances pointing to this similarity of great circles and straight lines; others will appear in the remaining paragraphs of Book IX.

**642.** It can be proved that the length of the arc of the great circle, less than a semicircle, between two points of a spherical surface is less than the length of any other curved line on the surface between the two points. Consequently, the *distance between two points on the surface of a sphere*, measured on the surface, is defined to be the arc of the great circle, less than a semicircle, drawn between them.

## Proposition II. Theorem

**643.** *All points in a circle of a sphere are equidistant from each of its poles.*




**Hypothesis.** $P$ and $P'$ are the poles of $\odot ABC$ of sphere $M$.

**Conclusion.** All points of $\odot ABC$ are equidistant from $P$, and also from $P'$.

**Proof.** 1. Let $A$ and $B$ be any two points of $\odot ABC$, and draw great circle arcs $PA$ and $PB$. Draw axis $PMP'$, intersecting the plane of $ABC$ at $O$. Draw $OA$, $OB$, $PA$, and $PB$.

2. $\therefore PA = PB$. Prove it.

3. $\therefore \overset{\frown}{PA} = \overset{\frown}{PB}$. Prove it.

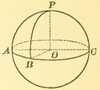
4. Since $A$ and $B$ are any two points of $\odot ABC$, $\therefore$ all points of $\odot ABC$ are equidistant from $P$.

5. Similarly all points of $\odot ABC$ are equidistant from $P'$.

**644.** The **Polar Distance** of a circle of a sphere is the distance from the nearer of its poles to the circle, or from either pole if they are equally near.

Thus, in the figure of Proposition II, the polar distance of $\odot ABC$ is arc $PA$.

**645. Cor.** *All points of a great circle of a sphere are at a quadrant's distance from either of its poles.*




**Note.** — The term *quadrant* in Spherical Geometry, usually signifies a quadrant of a great circle.

**Hypothesis.** $P$ is a pole of great circle $ABC$ of sphere $APC$; $B$ is any point in $\odot ABC$, and $PB$ is an arc of a great $\odot$.

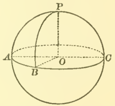
**Conclusion.** Arc $PB$ = a quadrant.

*Suggestion.* — Draw radii $OA$, $OB$, and $OP$.

**Note.** — An arc of a circle may be drawn on the surface of a sphere by placing one foot of the compasses at the nearer pole of the circle, the distance between the feet being equal to the chord of the polar distance.

## Proposition III. Theorem

**646.** *A point on the surface of a sphere at a quadrant's distance from each of two points, not the extremities of a diameter of the sphere, is a pole of the great circle through those points.*




**Hypothesis.** $P$ is on the surface of the sphere whose center is $O$. $AB$ is an arc of great $\odot ABC$, not a semicircle. $\widehat{PA}$ and $\widehat{PB}$ are quadrants.

**Conclusion.** $P$ is a pole of $\widehat{AB}$.

*Suggestions.* — 1. Recall the definition of "pole of a $\odot$."
2. Draw $PO$, $AO$, $OB$, and prove $PO \perp$ plane $ABC$.

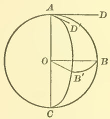
**Ex. 16.** If a point lies at a quadrant's distance from the ends of a diameter of a sphere, is it necessarily a pole of the great circle through those points?

**647.** The **angle between two intersecting curves** is the angle formed by the tangents to the curves at the point of intersection.

A **Spherical Angle** is the angle between two intersecting arcs of great circles.

## PROPOSITION IV. THEOREM

**648.** *A spherical angle is measured by an arc of a great circle having its vertex as a pole, included between its sides extended if necessary.*




**Hypothesis.** $ABC$ and $AB'C$ are arcs of great ⊚ on the surface of the sphere whose center is $O$; lines $AD$ and $AD'$ are tangent to $ABC$ and $AB'C$, respectively, and $BB'$ is an arc of a great ⊙ having $A$ as a pole, included between arcs $ABC$ and $AB'C$.

**Conclusion.** $\angle BAB'$ is measured by arc $BB'$.

**Proof.** 1. Draw diameter $AOC$ and radii $OB$ and $OB'$.

| | | |
|---|---|---|
| 2. | Arcs $AB$ and $AB'$ are quadrants. | Why? |
| 3. | $\therefore$ ∠s $AOB$ and $AOB'$ are rt. ∠s. | Why? |
| 4. | $OB \parallel AD$ and $OB' \parallel AD'$. | Why? |
| 5. | $\therefore \angle DAD' = \angle BOB'$. | § 481 |
| 6. But | $\angle BOB'$ is measured by arc $BB'$. | Why? |
| 7. Then, | $\angle DAD'$ is measured by arc $BB'$. | Why? |
| 8. | $\therefore \angle BAB'$ is measured by arc $BB'$. | § 647 |

**649. Cor. 1.** *The angle between two arcs of great circles is equal to the diedral angle formed by their planes.*

**650. Cor. 2.** *An arc of a great circle drawn to another great circle from the latter's pole is perpendicular to that great circle.*

*Suggestions.* — 1. What $\angle$ does $OA$ form with plane $BOB'$?
2. What kind of $\angle$ is diedral $\angle AOB'B$? (§ 495.)

**Ex. 17.** If a spherical blackboard can be had, construct a spherical angle and measure it.

**Ex. 18.** The chord of the polar distance of a circle of a sphere is 6. If the radius of the sphere is 5, what is the radius of the circle?

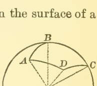
**Ex. 19.** What is the locus of points in space at a given distance $d$ from a fixed point, and equidistant from two given points?

## SPHERICAL POLYGONS

**Note.** — Recall at this point the definition of a polygon in plane geometry as a closed broken line lying in a plane. (§ 125.)

Recall also § 641, calling attention to the similarity in the rôles of the straight line in plane geometry and the great circle in solid geometry.

**651.** A **Spherical Polygon** is a closed line on the surface of a sphere consisting of arcs of three or more great circles; as polygon $ABCD$.




Just as in plane geometry we considered only convex polygons, so we shall consider only convex spherical polygons. (See § 126.)

It will be assumed as evident that a simple spherical polygon incloses a portion of the surface of the sphere. The bounding arcs are the **Sides** of the polygon; they are usually measured in arc-degrees. (§ 214.)

The angle formed by two consecutive sides of a polygon is an **Angle** of the spherical polygon, and its vertex is a **Vertex** of the polygon.

A **Diagonal** of a spherical polygon is the arc of the great circle joining two non-consecutive vertices of the polygon, and lying within the polygon.

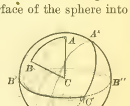
**652**. A **Spherical Triangle** is a spherical polygon having three sides. A spherical triangle is **Isosceles** when it has two equal sides; it is **Equilateral** when it has three equal sides; it is **Right-angled** when one of its angles is a right angle.

**653**. The planes of the sides of a spherical polygon form a polyedral angle, whose vertex is the center of the sphere, and whose face angles are measured by the sides of the spherical polygon.

Thus, in the figure of § 651, the planes of the sides of the spherical polygon form a polyedral, $\angle O\text{–}ABCD$, whose face angles $AOB$, $BOC$, etc., are measured by arcs $AB$, $BC$, etc., respectively.

**Ex. 20.** Prove that the angles of a spherical polygon have the same measures as the diedral angles of the corresponding polyedral angle.

**654**. If great circles be drawn with the vertices of a spherical triangle as poles, they divide the surface of the sphere into eight parts whose boundaries are triangles.




Thus, if circle $B'C'B''$ be drawn with vertex $A$ of spherical $\triangle ABC$ as a pole, circle $A'C'A''$ with $B$ as a pole, and circle $A'B''A''B'$ with $C$ as a pole, the surface of the sphere is divided into eight spherical $\triangle$s; namely, $A'B'C'$, $A'B''C'$, $A''B'C'$, and $A''B''C'$ on the hemisphere represented in the figure, and four others on the opposite hemisphere.

Of these eight spherical $\triangle$s, one is called the **Polar Triangle** of $ABC$, and is determined as follows:

Of the intersections, $A'$ and $A''$ of circles drawn with $B$ and $C$ as poles, that which is nearer to $A$, *i.e.* $A'$, is a vertex of the polar triangle; and similarly for the other intersections.

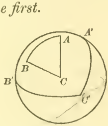
Thus, $A'B'C'$ is the polar $\triangle$ of $ABC$.

**Ex. 21.** The polar distance of a circle of a sphere is 60°. If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its center.

*Suggestion.* — Represent the radius of the sphere by $2x$. (§ 288.)

## Proposition V. Theorem

**655.** *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*




**Hypothesis.** $A'B'C'$ is the polar $\triangle$ of the spherical $\triangle\, ABC$; $A$, $B$, and $C$ are the poles of arcs $B'C'$, $C'A'$, and $A'B'$, respectively.

**Conclusion.** $ABC$ is the polar $\triangle$ of spherical $\triangle\, A'B'C'$.

**Proof.** 1. $A'$ is at a quadrant's distance from $B$. § 645
(Since $B$ is the pole of arc $A'C'$.)

2. $A'$ is at a quadrant's distance from $C$. Why?

3. $\therefore A'$ is a pole of the great $\odot$ arc $BC$. Why?

4. Similarly $B'$ and $C'$ are the poles of $\overset{\frown}{AC}$ and $\overset{\frown}{AB}$, respectively. Prove it.

5. $\therefore \triangle\, ABC$ is the polar $\triangle$ of $\triangle\, A'B'C'$.

(For of the two intersections of the great Ⓢ having $B'$ and $C'$, respectively, as poles, $A$ is nearer to $A'$; similarly for $B$ and $C$. § 654

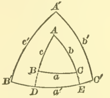
**Note.** — Two spherical triangles, each of which is the polar triangle of the other, are called polar triangles.

**Ex. 22.** How many degrees are there in the polar distance of a circle whose plane is $5\sqrt{2}$ units from the center of the sphere, the diameter of the sphere being 20 units?

*Suggestion.* — The radius of the $\odot$ is a leg of a rt. $\triangle$, whose hypotenuse is the radius of the sphere, and whose other leg is the distance from its center to the plane of the $\odot$.

## PROPOSITION VI. THEOREM

**656.** *In two polar triangles, each angle of one has the same measure as the supplement of that side of the other of which it is the pole.*




**Hypothesis.** $\triangle ABC$ and $A'B'C'$ are polar $\triangle$, point $A$ being the pole of $\widehat{B'C'}$, etc.

Let $a$, $a'$, etc. be the measures in degrees of $\widehat{BC}$, $\widehat{B'C'}$, etc., respectively; let $A$, $A'$, etc. be the measures in degrees of $\angle A$, $A'$, etc.

**Conclusion.** $A = 180 - a'$; $B = 180 - b'$; $C = 180 - c'$.
$A' = 180 - a$; $B' = 180 - b$; $C' = 180 - c$.

**Proof.** 1. Extend $\widehat{AB}$ and $\widehat{AC}$ to meet $\widehat{B'C'}$ at $D$ and $E$, respectively.

2. Since $B'$ is the pole of $\widehat{AE}$, $\widehat{B'E} = 90°$. Why?
3. Similarly, $\widehat{C'D} = 90°$.
4. $\therefore \widehat{B'E} + \widehat{C'D} = 180°$.
5. $\therefore \widehat{B'D} + \widehat{DE} + \widehat{C'D} = 180°$, or $\widehat{DE} + \widehat{B'C'} = 180°$.
6. But $\widehat{DE}$ is the measure of $\angle A$. § 648
7. $\therefore A + a' = 180$, or
$A = 180 - a'$.
8. Similarly for each of the other angles of either triangle.

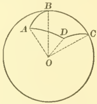
**Ex. 23.** Prove on the figure for § 656, that $A' = 180 - a$.

**Ex. 24.** If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

**Ex. 25.** If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

## Proposition VII. Theorem

**657.** *The sum of the sides of a convex spherical polygon is less than* 360°.



**Hypothesis.** $ABCD$ is a convex spherical polygon.

**Conclusion.** $\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360°$.

**Proof.** 1. Let polyedral angle $O\text{–}ABCD$ be the polyedral angle which corresponds to spherical polygon $ABCD$.

2. Then the measure of $\widehat{AB}$ equals the measure of central angle $AOB$. Why?

3. Similarly for arcs $BC$, $CD$, and $DA$.

4. But $\angle AOB + \angle BOC + \angle COD + \angle DOA < 360°$. § 514

5. $\therefore \widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360°$.

**Note 1.** — The pupil should recall at this point that one *arc-degree* is $\frac{1}{360}$ of *a* circle. Since arcs $AB$, $BC$, $CD$, and $DA$ are arcs of great circles, Proposition VII means that the sum of the sides of any spherical polygon is less than 360 arc-degrees of a great circle — i.e. *is less than a great circle*.

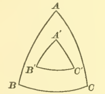
**Note 2.** — Proposition VII is one of many theorems about spherical polygons which can be formulated from corresponding theorems about polyedral angles by replacing in the latter the words "face angle" and "diedral angle" by "side" and "angle" respectively.

**Ex. 26.** If two sides of a spherical triangle measure 80° and 70° respectively, between what two values must the remaining side lie?

**Ex 27.** What is the greatest possible length for the sum of the sides of a convex spherical polygon on a circle of radius 12 inches?

## Proposition VIII. Theorem

**658.** *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°.




**Hypothesis.** $A$, $B$, and $C$ are the measures in degrees of the $\angle$s of spherical $\triangle ABC$.

**Conclusion.** $A+B+C > 180°$ and $< 540°$.

**Proof.** 1. Let $\triangle A'B'C'$ be the polar triangle of spherical $\triangle ABC$, $A$ being the pole of $\widehat{B'C'}$, $B$ of $\widehat{A'C'}$, and $C$ of $\widehat{A'B'}$.

Let the measures in degrees of $\widehat{B'C'}$, $\widehat{C'A'}$, and $\widehat{A'B'}$ be $a'$, $b'$, and $c'$, respectively.

| | | |
|---|---|---|
| 2. | $\therefore A = 180 - a'$, $B = 180 - b'$, and $C = 180 - c'$. | Why? |
| 3. | $\therefore A + B + C = 540 - (a' + b' + c')$. | Why? |
| 4. | $\therefore A + B + C < 540°$. | |
| 5. But | $a' + b' + c' < 360°$. | § 657 |
| 6. | $\therefore A + B + C > 180°$. | Ax. 20, § 158 |

**659. Cor.** *The sum of the angles of a spherical polygon of n sides is greater than* $(n-2) \times 180°$.

Consider $ABCD$ a spherical quadrilateral.

Draw great $\odot$ arc $BD$, dividing the quadrilateral into two spherical $\triangle$s, $ABD$ and $BDC$. The sum of the angles of the triangles equals the sum of the angles of the quadrilateral. In each triangle the sum of the angles is $> 180°$. Hence for the quadrilateral, the sum of the angles is greater than $2 \times 180°$; that is, the sum $> (4-2) \times 180°$.

In like manner, if there are $n$ sides, the polygon can be divided into $(n-2)$ spherical triangles, in each of which the sum of the angles is greater than 180°. Therefore, in the polygon the sum of the angles is greater than $(n-2) \times 180°$.

**660.** The **Spherical Excess** of a spherical triangle, measured in degrees, is the difference between the sum of its angles and 180°.

The **Spherical Excess** of a spherical polygon of $n$ sides, measured in degrees, is the difference between the sum of its angles and $(n - 2) \times 180°$.

Note.— In each case, the spherical excess is the amount by which the sum of the angles of the spherical polygon exceeds the sum of the angles of a plane polygon of the same number of sides.

**Ex. 28.** Prove that a spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.

**661.** A spherical triangle having two right angles is called a **Bi-rectangular Triangle**, and one having three right angles a **Tri-rectangular Triangle.**

**Ex. 29.** Prove that the sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles.

**Ex. 30.** What is the spherical excess of a triangle whose angles are 100°, 95°, and 65°, respectively?

**Ex. 31.** What is the spherical excess of a tri-rectangular triangle?

**Ex. 32.** Prove that the spherical excess of a bi-rectangular triangle is the measure of the remaining angle of the triangle. (§ 648.)

**Ex. 33.** What is the spherical excess of a triangle if the sides of its polar triangle measure 80°, 85°, and 95°?

**Ex. 34.** What relation exists between a tri-rectangular spherical triangle and its polar?

**Ex. 35.** Prove that the sides opposite the equal angles of a bi-rectangular triangle are quadrants.

*Suggestion.*— Recall § 499 and the definition of "pole."

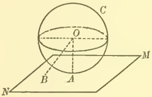
**Ex. 36.** Prove that each side of a tri-rectangular triangle is a quadrant.

**Ex. 37.** Prove that in a bi-rectangular spherical triangle, the third angle has the same measure as the side opposite it.

**662.** If a plane or a line has only one point in common with the surface of a sphere, it is said to be **Tangent to the Sphere.** The sphere is said to be tangent to the plane or line. The point common to the plane or line and the spherical surface is called the **Point of Contact or Tangency.**

## Proposition IX. Theorem

**663.** *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*




**Hypothesis.** Plane $MN \perp$ radius $OA$ at $A$.

**Conclusion.** Plane $MN$ is tangent to the sphere.

*Suggestion.* — Let $B$ be any point of $MN$ except $A$. Prove that $B$ lies outside the sphere.

**664. Cor.** *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (Fig. of Prop. IX.)

**665.** Two **Spheres** are **Tangent** to each other when each is tangent to the same plane at the same point.

**Ex. 38.** Prove that a straight line perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.

**Ex. 39.** How many straight lines can be tangent to a sphere at a point of the sphere?

**Ex. 40.** If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

**Ex. 41.** Prove that all lines tangent to a sphere at a point of the sphere lie in the plane tangent to the sphere at that point.

**Ex. 42.** How many straight lines can be tangent to a sphere from a point outside the sphere? Compare the lengths of these tangents.

**Ex. 43.** Prove that the points of contact of all lines tangent to a sphere from an exterior point lie in a circle.

**Ex. 44.** Is the circle in Ex. 43 a great circle or a small circle?

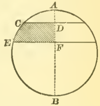
**Ex. 45.** If two spheres are tangent to the same plane at the same point, the straight line joining their centers passes through the point of contact.

### MEASUREMENT OF SPHERICAL POLYGONS

**666.** A **Zone** is the portion of a spherical surface included between two parallel planes.

The circles which bound the zone are its *bases*, and the distance between their planes is its *altitude*.

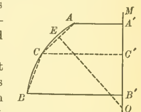
A *zone of one base* is a zone lying between one plane and a parallel plane tangent to the sphere.




**667.** If semicircle $ACEB$ be revolved about diameter $AB$ as an axis, and $CD$ and $EF$ are lines $\perp AB$, then arc $CE$ generates a zone whose altitude is $DF$, and arc $AC$ a zone of one base whose altitude is $AD$.

**668. Application of Limits to Zones.**

Let $O$ be the center of $\widehat{ACB}$, and $OM$ be any diameter of the circle. Let $AA'$ and $BB'$ be $\perp OM$. Let $C$ bisect arc $AB$, and draw broken line $ACB$. If arc $AB$ is revolved about $OM$ as axis, it generates a zone, and broken lines $AC$ and $CB$ generate curved surfaces.

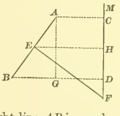



It will be assumed as evident that the sum of the areas of the surfaces generated by $AC$ and $CB$ is less than the area of the zone generated by $\widehat{ACB}$.

Assume arcs $AC$ and $CB$ to be bisected at $X$ and $Y$, and imagine the broken line $AXCYB$. It will be assumed as evident that the area of the surface generated by line $AXCYB$ is greater than that generated by $ACB$ but is still less than that of the zone generated by $\widehat{AB}$. If the process of subdividing arc $AB$ by successively halving the subdivisions of arc $AB$ be continued indefinitely, it will be assumed evident that the surface generated by the resulting broken line approaches as limit the zone $AB$; also, as the chords like $AC$ decrease indefinitely in length, their distance from center $O$ increases and approaches the radius of arc $AB$ as limit.

## Proposition X. Theorem

**669.** *The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the mid-point of the line and terminating in the axis.*




**Hypothesis.** Straight line $AB$ is revolved about straight line $FM$ in its plane, not $\perp$ to and not intersecting it, as an axis; lines $AC$ and $BD \perp FM$, and $EF$ is the $\perp$ erected at the mid-point of $AB$ terminating in $FM$.

**Conclusion.** Area $AB^* = CD \times 2\pi EF$.

**Proof.** 1. Draw line $AG \perp BD$, and line $EH \perp CD$.

2. The surface generated by $AB$ is the lateral surface of a frustum of a cone of revolution, whose bases are generated by $AC$ and $BD$.

3. $\therefore$ area $AB = AB \times 2\pi EH$ — § 619

4. $\triangle ABG$ and $EFH$ are similar. — Prove it.

5. $\therefore \dfrac{AB}{AG} = \dfrac{EF}{EH}$. — Why?

6. $\therefore AB \times EH = AG \times EF$ — Why?

$= CD \times EF$. — Why?

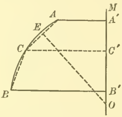
7. Substituting in Step 3,

area $AB = CD \times 2\pi EF$.

*The expression "area $AB$" is used to denote the area of the surface generated by $AB$.

## Proposition XI. Theorem

**670.** *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*




**Hypothesis.** $\overset{\frown}{AB}$ is revolved about diameter $OM$ as axis; $AA'$ and $BB' \perp OM$; $R$ is the radius of $\overset{\frown}{AB}$.

**Conclusion.** Area of zone generated by $\overset{\frown}{AB} = A'B' \times 2\,\pi R$.

**Proof.** 1. Bisect $\overset{\frown}{AB}$ at $C$; draw $AC$ and $CB$; also draw $CC' \perp OM$ and $OE \perp AC$.

2. Revolve the figure about $OM$ as axis.

3. $\therefore$ area $AC = A'C' \times 2\,\pi OE$. § 669

area $CB = C'B' \times 2\,\pi OE$. Why?

4. Adding, the surface generated by $ACB$

$$= (A'C' + C'B') \times 2\,\pi OE.$$

5. Continue to bisect the subdivisions of $\overset{\frown}{AB}$, indefinitely.

6. Then, the area of the surface generated by revolving the inscribed broken line approaches as limit the area of the zone generated by $\overset{\frown}{AB}$, and $OE \doteq R$. § 668

7. $\therefore$ area of zone $= A'B' \times 2\,\pi R$. § 403

**Note.** — The proof of § 670 holds for any zone which lies entirely on the surface of a hemisphere; for, in that case, no chord is $\parallel$ $OM$, and § 669 is applicable.

Since a zone which does not lie entirely on the surface of a hemisphere may be considered as the sum of two zones, each of which does lie entirely on the surface of a hemisphere, the theorem of § 670 is true for any zone.

**671. Cor. 1.** *If $Z$ denotes the area of a zone, $h$ its altitude, and $R$ the radius of the sphere,*

$$Z = 2\,\pi Rh.$$

**672. Cor. 2.** *The area of a spherical surface equals the square of its radius multiplied by $4\,\pi$.*

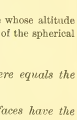
**Proof.** A spherical surface may be regarded as a zone whose altitude is a diameter of the sphere. Letting $S$ represent the area of the spherical surface,

$$S = 2\,\pi R \cdot 2\,R = 4\,\pi R^2.$$

**673. Cor. 3.** *The area of the surface of a sphere equals the area of four great circles of the sphere.*

**674. Cor. 4.** *The areas of two spherical surfaces have the same ratio as the squares of their radii or the squares of their diameters.*

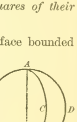
**675.** A **Lune** is the portion of a spherical surface bounded by two semicircles of great circles; as $ACBD$.




The **Angle of a Lune** is the angle between its bounding arcs.

It is evident that two lunes on the same sphere or equal spheres are congruent if their angles are equal.

**676.** It is evident that lunes on the same sphere or on equal spheres may be added by placing them so that their angles become adjacent angles; thus lune $ACBE$ + lune $AEBF$ = lune $ACBF$.




When two lunes are added, the angle of the sum equals the sum of the angles of the given lunes.

If $L_X$ is used to denote the lune whose angle is $\angle X$, then

$$L_X + L_Y = L_{(X+Y)};$$

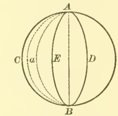
*i.e.* lune of $\angle X$ + lune of $\angle Y$ = lune of $\angle(X + Y)$.

**Note.** — It is very important that the symbol $L_X$ be understood and remembered.

## Proposition XII. Theorem

**677.** *Two lunes on the same sphere or equal spheres have the same ratio as their angles.*

Case I. *When the angles are commensurable.*




**Hypothesis.** $ACBD$ and $ACBE$ are lunes on sphere $AB$, having their $\angle s$ $CAD$ and $CAE$ commensurable.

**Conclusion.** $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

**Proof.** 1. Let $\angle CAa$ be a common measure of $\angle CAD$ and $CAE$, and let it be contained 5 times in $\angle CAD$, and 3 times in $\angle CAE$.

2. $$\therefore \frac{\angle CAD}{\angle CAE} = \frac{5}{3}. \qquad (1)$$

3. Extending the arcs of division of $\angle CAD$ to $B$, lune $ACBD$ will be divided into 5 parts, and lune $ACBE$ into 3 parts, all of which parts will be equal. Why?

4. $$\therefore \frac{ACBD}{ACBE} = \frac{5}{3}. \qquad (2)$$

5. From (1) and (2), $\dfrac{ACBD}{ACBE} = \dfrac{\angle CAD}{\angle CAE}$. Why?

**Note.** — The theorem may be proved in a similar manner when the given lunes are on equal spheres.

Case II. *When the angles are incommensurable.*

*Suggestion.* — Model the proof after that in § 544.

**678.** The surface of a hemisphere may be regarded as a lune of angle 180° and the surface of the sphere, a lune of angle 360°.

**679. Cor. 1.** *The surface of a lune is to the surface of the sphere as the measure of its angle in degrees is to* 360.

**680. Cor 2.** *If the radius of the sphere is $R$, and the angle of the lune, measured in degrees, is $A$, and the area of the lune is denoted by $L_A$, then*

$$L_A = \frac{A}{360} \times 4\pi R^2 = \frac{\pi R^2 A}{90}.$$

**Ex. 46.** Prove that the areas of two zones on the same sphere, or equal spheres, are to each other as their altitudes.

**Ex. 47.** Determine the area of a zone whose altitude is 13, if the radius of the sphere is 16.

**Ex. 48.** Prove that the area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc. (§ 288.)

**Ex. 49.** Determine the area of the surface of a sphere whose radius is 12.

**Ex. 50.** If the radius of a sphere is $R$, what is the area of a zone of one base, whose generating arc is 45°?

**Ex. 51.** Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$, and radius of base 3.

**Ex. 52.** What is the area of a lune whose angle is 40° on the surface of a sphere whose radius is 15 in.?

**Ex. 53.** What part of the surface of the earth is included between the 30th and 35th meridians?

**Ex. 54.** The area of a lune is $28\frac{4}{5}$. If the area of the surface of the sphere is 120, what is the angle of the lune?

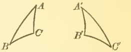
**Ex. 55.** Prove that the surface of a sphere is equal to two thirds the entire surface of the right circular cylinder circumscribed about it.

**Ex. 56.** Compare the surface of a sphere with the lateral surface of the right circular cylinder circumscribed about the sphere.

**Ex. 57.** What circles of the earth bound the North Temperate Zone? What part of the earth's surface lies within that zone?

**Ex. 58.** What zones of the earth are zones of one base?

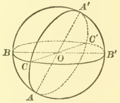
**681.** Two spherical polygons, on the same or equal spheres, are **Symmetrical** when the sides and angles of one are equal, respectively, to the sides and angles of the other, if the equal parts occur in opposite orders.




Thus, if spherical ▲ $ABC$ and $A'B'C'$, on the same or equal spheres, have sides $AB$, $BC$, and $CA$ equal, respectively, to sides $A'B'$, $B'C'$, and $C'A'$, and ∠s $A$, $B$, and $C$ to ∠s $A'$, $B'$, and $C'$, and the equal parts occur in the opposite orders the ▲ are symmetrical.

## Proposition XIII. Theorem

**682.** *The spherical triangles corresponding to a pair of vertical triedral angles are symmetrical.*




**Hypothesis.** $AOA'$, $BOB'$, and $COC'$ are diameters of the sphere with center $O$; the planes determined by them intersect the spherical surface in ⊙ $ABA'B'$, $ACA'C'$, and $BCB'C'$.

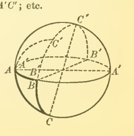
**Conclusion.** Spherical ▲ $ABC$ and $A'B'C'$ are symmetrical.

*Suggestions.*—1. Prove $\widehat{A'B'} = \widehat{AB}$; $\widehat{B'C'} = \widehat{BC}$, etc.

2. Prove $\angle BCA = \angle B'C'A'$; $\angle BAC = \angle B'A'C'$; etc.

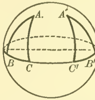
3. Prove that the parts of $\triangle ABC$ occur in opposite order to those of $\triangle A'B'C'$.

The adjoining figure will aid in doing this. $A'B'C'$ has been slid around the sphere until it occupies the position indicated in this figure. Determine the direction from $A$ to $B$ to $C$; and also the direction from $A'$ to $B'$ to $C'$.



## PROPOSITION XIV. THEOREM

**683.** *Two symmetrical spherical triangles, of which one is isosceles, are congruent and hence equal.*



**Hypothesis.** $\triangle ABC$ is symmetrical to $\triangle A'B'C'$; that is, $\widehat{AB} = \widehat{A'B'}$, $\widehat{AC} = \widehat{A'C'}$, $\widehat{BC} = \widehat{B'C'}$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle A = \angle A'$, with the equal parts arranged in opposite orders in the triangles; also $\widehat{AB} = \widehat{AC}$.

**Conclusion.** $\triangle ABC \cong \triangle A'B'C'$.

**Proof.** 1. Since $\widehat{AB} = \widehat{A'B'}$, and $\widehat{AB} = \widehat{AC}$,

$$\therefore \widehat{A'B'} = \widehat{AC}.$$

2. In like manner $\widehat{A'C'} = \widehat{AB}$.

Complete the proof by superposing $\triangle A'B'C'$ on $\triangle ABC$, making $\widehat{A'C'}$ coincide with $\widehat{AB}$, with point $A'$ on point $A$.

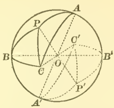
**684. Cor.** *If one of two symmetrical spherical triangles is isosceles, the other is also.*

## PROPOSITION XV. THEOREM

**685.** *Two spherical triangles corresponding to a pair of vertical triedral angles are equal.*

**Hypothesis.** $AOA'$, $BOB'$, and $COC'$ are diameters of sphere $O$; also, the planes determined by them intersect the surface in arcs $AB$, $BC$, $AC$, $A'B'$, $B'C'$, and $A'C'$. (Fig. p. 431.)

**Conclusion.** Area of $\triangle ABC$ = area of triangle $A'B'C'$.




**Proof.** 1. Let $P$ be the pole of the small circle passing through points $A$, $B$, and $C$; draw arcs of great circles $PA$, $PB$, and $PC$.

2. $\therefore \widehat{PA} = \widehat{PB} = \widehat{PC}$. § 643

3. Draw $PP'$, a diameter of the sphere, and $P'A'$, $P'B'$, and $P'C'$ arcs of great ⓢ; then spherical $\triangle$s $PAB$ and $P'A'B'$ are symmetrical. § 682

4. But spherical $\triangle PAB$ is isosceles.

5. $\therefore \triangle PAB = \triangle P'A'B'$. Why?

6. In like manner,

$$\triangle PBC = \triangle P'B'C' \text{ and } \triangle PCA = \triangle P'C'A'.$$

7. Then the sum of the areas of $\triangle$s $PAB$, $PBC$, $PAC$ equals the sum of the areas of $\triangle$s $P'A'B'$, $P'B'C'$, and $P'C'A'$.

8. $\therefore$ area $\triangle ABC$ = area $\triangle A'B'C'$.

**686. Cor.** *Two symmetrical triangles on the same or equal spheres are equal.*

1. Let $\triangle A''B''C''$ be symmetrical to $\triangle ABC$; let $\triangle A'B'C'$ be the spherical triangle on the same sphere as $\triangle ABC$, such that $A'$, $B'$, and $C'$ are diametrically opposite to $A$, $B$, and $C$, respectively.

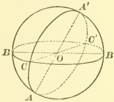
2. Then $\triangle A'B'C'$ is also symmetrical to $\triangle ABC$, and equal to $\triangle ABC$.

3. $\therefore$ the parts of $\triangle A''B''C''$ and $\triangle A'B'C'$ are equal and are arranged in the same order. Hence $\triangle A''B''C'' \cong A'BC''$.

4. $\therefore \triangle ABC = \triangle A''B''C''$.

## Proposition XVI. Theorem

**687.** *A spherical triangle of a sphere equals one half a lune of that sphere whose angle in degrees equals the spherical excess of the triangle.*




**Hypothesis.** $A$, $B$, and $C$ are the measures in degrees of the angles of spherical $\triangle ABC$. $E$ represents the spherical excess of the triangle.

**Conclusion.** $\triangle ABC = \frac{1}{2}L_E$;

that is, $\triangle ABC =$ one half a lune whose angle is $E$.

**Proof.** 1. Complete the ⊚ $ABA'B'$, $BCB'C'$, and $ACA'C'$, and draw diameters $AA'$, $BB'$, and $CC'$.

2. $\triangle ABC + \triangle ACB' =$ lune of $\angle B$, or $L_B$.

3. $\triangle ABC + \triangle A'CB =$ lune of $\angle A$, or $L_A$.

4. $\triangle ABC + \triangle ABC' =$ lune of $\angle C$, or $L_C$.

But $\triangle A'B'C = \triangle ABC'$, so § 685

5. $\triangle ABC + \triangle A'B'C =$ lune of $\angle C$, or $L_C$.

6. Adding the equations of steps 2, 3, and 5,

$2\triangle ABC + (\triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'B'C)$
$= L_{(A+B+C)}$. § 676

7. But $\triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'B'C$
= surface of hemisphere
$= L_{180}$. § 678

8. $\therefore 2\triangle ABC + L_{180} = L_{A+B+C}$.

9. $\therefore 2\triangle ABC = L_{A+B+C} - L_{180} = L_{(A+B+C)-180}$.

10. But $A + B + C - 180 = E$.

11. $\therefore 2\triangle ABC = L_E$, or $\triangle ABC = \frac{1}{2}L_E$.

**688. Cor. 1.** *If the radius of the sphere is $R$ and the spherical excess of $\triangle ABC$ in degrees is $E$, then*

$$\text{area of } \triangle ABC = \tfrac{1}{2} L_E = \tfrac{1}{2}\frac{\pi R^2 E}{90} = \frac{\pi R^2 E}{180}.$$

**689. Cor. 2.** *The area of any spherical polygon whose excess is $E$ is $\dfrac{\pi R^2 E}{180}$.*

*Suggestion.* — Divide the polygon into $\triangle$s by drawing diagonals from one vertex. Express the area of each triangle. Remember that the excess of the polygon equals the sum of the excesses of the triangles, when the polygon is divided as suggested.

**Note.** — A *spherical degree* may be defined as being a bi-rectangular spherical triangle whose third angle is one spherical angular degree. The area of a spherical triangle in spherical degrees can be proved to equal its spherical excess in degrees.

**Ex. 59.** Determine the area of a spherical triangle whose angles are 125°, 133°, and 156°, on a sphere whose radius is 10 in.

**Ex. 60.** What is the ratio of the areas of two spherical triangles on the same sphere whose angles are 94°, 135°, and 146°, and 87°, 105°, and 118°, respectively.

**Ex. 61.** Determine the area of a spherical triangle whose angles are 103°, 112°, and 127° on a sphere whose area is 160.

**Ex. 62.** Find the area of a spherical hexagon whose angles are 120°, 139°, 148°, 155°, 162°, and 167°, on a sphere whose radius is 12.

**Ex. 63.** The sides of a spherical triangle on a sphere whose radius is 15 in. are 44°, 63°, and 97°. Find the area of its polar triangle.

**Ex. 64.** Determine the part of the area of the surface of a sphere intercepted by a triedral angle whose face angles are 89°, 55°, and 100°.

**Ex. 65.** Express the ratio of a spherical triangle to the surface of the sphere in terms of the spherical excess $E$ of the triangle, when ($a$) the excess is measured in degrees; ($b$) the excess is measured in right angles.

**Ex. 66.** The area of a spherical pentagon, four of whose angles are 112°, 131°, 138°, and 168°, is 27. If the area of the surface of the sphere is 120, what is the other angle?

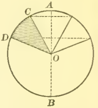
**Ex. 67.** What part of the surface of a sphere is a tri-rectangular triangle of the sphere?

**Ex. 68.** Compare the area of a tri-rectangular spherical triangle of a sphere whose radius is 10 in. with the area of the plane triangle formed by the chords of the sides of the spherical triangle.

### VOLUME OF A SPHERE

**690.** If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a **Spherical Sector.**

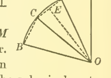
Thus if semicircle $ACDB$ be revolved about diameter $AB$ as an axis, sector $OCD$ generates a spherical sector.




The zone generated by the arc of the circular sector is called the base of the spherical sector.

Note. — In the following pages, the expression "Vol. $OCD$" will be used to denote the volume of the solid generated by revolving the portion of the plane within $OCD$ around some axis specified.

**691. Application of Limits to Spherical Sectors.** Let $O$ be the center of arc $AB$, and $OM$ be any diameter of the circle whose center is $O$. Let $C$ bisect arc $AB$, and draw radii $OA$, $OB$, and $OC$. Draw $OE \perp AC$; draw broken line $ACB$.



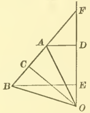

If the sector $OAB$ is revolved about $OM$ as an axis, it generates a spherical sector. The portion of the plane bounded by polygon $OACB$ generates a solid which is less than the spherical sector generated by circular sector $OAB$.

If arcs $AC$ and $CB$ are bisected at $D$ and $F$ respectively, and broken line $ADCFB$ is drawn, then when the figure is revolved about $OM$, the part of the plane bounded by polygon $OADCFB$ generates a solid more nearly equal to the spherical sector. If the process of halving the arcs be continued indefinitely, it will be assumed evident that the solid generated by the part of the plane bounded by the inscribed polygon approaches the spherical sector as limit.

Notice that the surface generated by broken line $ADCFB$ approaches as limit the zone generated by arc $ACB$. (§ 668.)

## Proposition XVII. Theorem

**692.** *If a portion of a plane inclosed by an isosceles triangle be revolved about a straight line in its plane as axis, which passes through its vertex without intersecting its surface and without being parallel to its base, the volume of the solid generated is equal to the area of the surface generated by its base multiplied by one third its altitude.*




**Hypothesis.** Isosceles $\triangle OAB$, and the surface inclosed, are revolved about straight line $OF$ in its plane; $OF$ is not $\parallel$ base $AB$; $OC \perp AB$.

**Conclusion.** Vol. $OAB =$ area $AB \times \frac{1}{3} OC$.

**Proof.** 1. Draw $AD \perp OF$ and $BE \perp OF$; extend $BA$ to meet $OF$ at $F$.

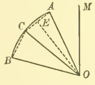
2. Vol. $OBF =$ vol. $OBE +$ vol. $BEF$
3. $= \frac{1}{3}\pi \overline{BE}^2 \times OE + \frac{1}{3}\pi \overline{BE}^2 \times EF$ § 616
4. $= \frac{1}{3}\pi \overline{BE}^2(OE + EF) = \frac{1}{3}\pi BE \times BE \times OF.$
5. But $BE \times OF = OC \times BF.$ Prove it.
6. $\therefore$ vol. $OBF = \frac{1}{3}\pi BE \times OC \times BF.$
7. But $\pi BE \times BF$ is the area $BF$. § 614
8. $\therefore$ vol. $OBF =$ area $BF \times \frac{1}{3} OC.$
9. Similarly vol. $OAF =$ area $AF \times \frac{1}{3} OC.$
10. Subtracting 9 from 8,

vol. $OAB =$ (area $BF -$ area $AF) \times \frac{1}{3} OC.$

11. $=$ area $AB \times \frac{1}{3} OC.$

## Proposition XVIII. Theorem

**693.** *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one third the radius of the sphere.*




**Hypothesis.** Sector $OAB$ of $\odot O$ is revolved about diameter $OM$ as an axis; $R$ is the radius of the sphere.

**Conclusion.** Vol. of the spherical sector generated by circular sector $OAB =$ area of zone generated by $\widehat{AB} \times \frac{1}{3} R$.

**Proof.** 1. Let $C$ bisect $\widehat{AB}$. Draw $AC$, $CB$, $OC$; and draw $OE \perp AC$.

2. Vol. $OAC =$ area $AC \times \frac{1}{3} OE$. § 691

3. Vol. $OCB =$ area $CB \times \frac{1}{3} OE$.

4. Adding, vol. $OACB =$ (area $AC +$ area $CB$) $\times \frac{1}{3} OE$
$= $ area $ACB \times \frac{1}{3} OE$.

5. Let the subdivisions of $\widehat{AB}$ be bisected indefinitely.

6. Then vol. $OACB \doteq$ vol. generated by sector $OAB$,
and area $ACB \doteq$ area of zone generated by $\widehat{AB}$
and $OE = R$.

7. $\therefore$ vol. generated by sector $OAB$
$=$ area of zone generated by $\widehat{AB} \times \frac{1}{3} R$. § 403

**694. Cor. 1.** *If $v$ denotes the volume of a spherical sector, $h$ the altitude of the zone which forms its base, and $R$ the radius of the sphere,*

$$v = 2\,\pi Rh \times \tfrac{1}{3} R = \tfrac{2}{3}\pi R^2 h. \qquad \S\ 671$$

**695. Cor. 2.** The sphere may be regarded as a spherical sector whose base is the entire surface of the sphere. Letting

$V$ denote the volume of the sphere, and $R$ its radius,

$$V = 4\,\pi R^2 \times \tfrac{1}{3}\,R = \tfrac{4}{3}\,\pi R^3.$$

*The volume of a sphere is equal to the cube of its radius multiplied by $\frac{4}{3}\pi$.*

**696. Cor.** 3. *The volumes of two spheres have the same ratio as the cubes of their radii.*

**Ex. 69.** Find the volume of the sphere whose radius is 12.

**Ex. 70.** Determine the volume of metal in a spherical shell 10 in. in diameter and 1 in. thick.

**Ex. 71.** A spherical cannon ball 9 in. in diameter is dropped into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box?

**Ex. 72.** If a sphere 6 in. in diameter weighs 351 oz., what is the weight of a sphere of the same material whose diameter is 10 in.?

**Ex. 73.** The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is the weight, if a cubic inch of the metal weighs one third pound?

**Ex. 74.** Find the area of the surface and the volume of the sphere inscribed in a cube the area of whose surface is 486 sq. in.

**Ex. 75.** Find the radius and the volume of a sphere, the area of whose surface is $324\,\pi$ sq. in.

**Ex. 76.** Prove that the volume of a sphere is two thirds the volume of its circumscribed cylinder.

**Ex. 77.** Within a sphere of radius $R$ is inscribed a right circular cylinder whose altitude equals the diameter of its base.

($a$) Determine its lateral area and compare the result with the area of the surface of the sphere.

($b$) Compare its volume with the volume of the sphere.

**Ex. 78.** Given a spherical surface of radius $R$ and its circumscribed right circular cylinder. From the center of the sphere, draw lines to the points of the circles bounding the bases of the cylinder, thus forming the two right circular cones. Compare the volume of the sphere with the difference between the volume of the cylinder and the sum of the volumes of the two cones.

**Ex. 79.** A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

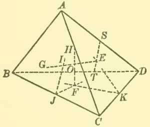
**Note.** — Supplementary Exercises 72–85, p. 460, can be studied now.

SUPPLEMENTARY TOPICS

## Group A. Construction of Spheres

### Proposition XIX. Theorem

**697.** *Through four points not in the same plane, one and only one spherical surface can be passed.*



**Hypothesis.** $A$, $B$, $C$, and $D$ are four points, not in the same plane.

**Conclusion.** One and only one spherical surface can be passed through $A$, $B$, $C$, and $D$.

**Proof.** 1. Every point equidistant from $C$ and $D$ must lie in a plane $EKF \perp CD$ at its mid-point $K$; and conversely.
§§ 457; 459

2. Every point equidistant from $B$ and $C$ must lie in a plane $IJF \perp BC$ at its mid-point $J$; and conversely.

3. These planes intersect in a line $HF$, which is $\perp$ plane $BCD$ at $F$, the circumcenter of $\triangle BCD$. § 499

Also, by steps 1 and 2, every point equidistant from $B$, $C$, and $D$ must lie in $HF$; and conversely.

4. Similarly, every point equidistant from $A$, $C$, and $D$ lies in line $GE$, which is $\perp$ plane $ACD$ at $E$, the circumcenter of $\triangle ACD$.

5. Since $E$ and $F$ are in plane $EKF$, then $GE$ and $HF$ lie in plane $EKF$. § 497

6. $\therefore GE$ intersects $HF$ at a point $O$.

7. $\therefore O$ is one and the only point which is equidistant from $A$, $B$, $C$, and $D$. Steps 3 and 4

8. $\therefore$ a sphere with center $O$ and radius $OA$ is the one and only sphere through $A$, $B$, $C$, and $D$.

**698. Cor.** *A sphere may be circumscribed about any tetraedron.*

**Ex. 80.** What is the locus of the center of a sphere which will have a given radius $r$ and will pass through a given point $P$?

**Ex. 81.** What is the locus of the center of a sphere which will pass through each of two given points?

**Ex. 82.** What is the locus of the center of a sphere which will pass through each of three given points?

**Ex. 83.** What is the locus of the center of a sphere which will pass through all the points of a given circle?

**Ex. 84.** Is it possible for a sphere to pass through all the points of a given circle and also through a given point outside the plane of the circle? If so, tell how to determine its center and its radius.

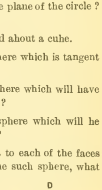
**Ex. 85.** Prove that a sphere can he circumscribed ahout a cuhe.

**Ex. 86.** What is the locus of the center of a sphere which is tangent to a given plane at a given point?

**Ex. 87.** What is the locus of the center of a sphere which will have a given radius $r$ and will he tangent to a given plane?

**Ex. 88.** What is the locus of the center of a sphere which will he tangent to each of the faces of a given diedral angle?

**Ex. 89.** Is it possihle for a sphere to be tangent to each of the faces of a given triedral angle? If there is more than one such sphere, what is the locus of the center?




**Ex. 90.** Find the area of the spherical surface passing through the vertices of a regular tetraedron whose edge is 8.

*Suggestion.* — Draw $DOE$ and $AOF$ perpendicular to $\triangle ABC$ and $BCD$ respectively.

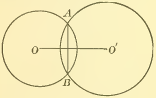
**Ex. 91.** Prove that a sphere can he inscribed in a given tetraedron.

## Group B. General Theorems of Spherical Geometry

**699.** Special interest attaches to the following theorems because of their similarity to certain theorems of plane geometry. In each case, the pupil should recall the corresponding theorem of plane geometry, if there is one, or should note the difference between the theorem of spherical geometry and the corresponding situation in plane geometry.

### Proposition XX. Theorem

**700.** *The intersection of two spherical surfaces is a circle, whose center is in a straight line joining the centers of the spheres and whose plane is perpendicular to that line.*




**Hypothesis.** $O$ and $O'$ are the centers of two intersecting spherical surfaces.

**Conclusion.** The intersection of the surfaces is a circle whose center is in line $OO'$ and whose plane is $\perp$ $OO'$.

**Proof.** 1. Through $O$ and $O'$ and any point $A$ of the intersection, pass a plane. This plane cuts the two surfaces in two intersecting great Ⓢ. Let $AB$ be the common chord of these two Ⓢ, intersecting $OO'$ at $C$.

2. $\therefore$ $OO'$ bisects $AB$ at right angles. § 207

3. If the entire figure is revolved about $OO'$ as an axis, the Ⓢ will generate the spherical surfaces whose centers are $O$ and $O'$.

Point $A$ will generate a ⊙ whose center is $C$ and radius $AC$, which is common to the two spherical surfaces.

4. The plane of ⊙ $AC$ is $\perp$ $OO'$. § 458

5. No point outside $\odot\ ACB$ can lie in both surfaces; for, if there were such a point, the two surfaces would necessarily coincide. § 697

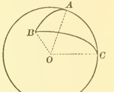
**Ex. 92.** What is the locus of points at the distance $r_1$ from a given point $P_1$ and at the distance $r_2$ from a given point $P_2$?

**Ex. 93.** The distance between the centers of two spheres whose radii are 25 and 17, respectively, is 28. Find the diameter of their circle of intersection, and its distance from the center of each sphere.

*Suggestion.* — Recall § 313.

## Proposition XXI. Theorem

**701.** *Any side of a spherical triangle is less than the sum of the other two.*




**Hypothesis.** $AB$ is any side of spherical $\triangle\ ABC$.

**Conclusion.** $\widehat{AB} < \widehat{AC} + \widehat{BC}$.

*Suggestions.* — 1. Compare $\angle\ AOB$ with $\angle\ AOC + \angle\ BOC$.

2. What is the relation of the measure of $\widehat{AB}$ and that of $\angle\ AOB$? Of $\widehat{AC}$ and $\angle\ AOC$? etc.

**Ex. 94.** Prove that any side of a convex spherical polygon is less than the sum of the remaining sides.

**Ex. 95.** Prove that the sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

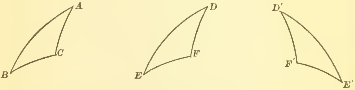
**702.** Two spherical polygons on the same sphere or equal spheres are mutually equilateral or mutually equiangular when the sides or angles of one are equal respectively to the sides or angles of the other, whether taken in the same or in opposite orders.

## Proposition XXII. Theorem

**703.** *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal respectively to two sides and the included angle of the other,*

I. *They are congruent if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in opposite orders.*




I. **Hypothesis.** $ABC$ and $DEF$ are spherical $\triangle$s on the same sphere, or equal spheres, having $\widehat{AB} = \widehat{DE}$, $\widehat{AC} = \widehat{DF}$, and $\angle A = \angle D$; and the equal parts occur in the same order.

**Conclusion.** $\triangle ABC \cong \triangle DEF.$

*Suggestion.* — Prove it by superposition as in § 63.

II. **Hypothesis.** $ABC$ and $D'E'F'$ are spherical $\triangle$s on the same sphere, or equal spheres, having $\widehat{AB} = \widehat{D'E'}$, $\widehat{AC} = \widehat{D'F'}$, and $\angle A = \angle D'$; and the equal parts occur in opposite orders.

**Conclusion.** $ABC$ and $D'E'F'$ are symmetrical $\triangle$s.

**Proof.** 1. Let $DEF$ be a spherical $\triangle$ on the same sphere, or an equal sphere, symmetrical to $\triangle D'E'F'$, having

$$\widehat{DE} = \widehat{D'E'},\quad \widehat{DF} = \widehat{D'F'}, \text{ and } \angle D = \angle D',$$

the equal parts occurring in opposite orders.

2. Then in spherical $\triangle$s $ABC$ and $DEF$,

$$\widehat{AB} = \widehat{DE},\ \widehat{AC} = \widehat{DF}, \text{ and } \angle A = \angle D;$$

and the equal parts occur in the same order.

3. $\therefore \triangle ABC \cong \triangle DEF.$

4. $\therefore \triangle ABC$ is symmetrical to $\triangle D'E'F'$.

**Ex. 96.** Prove that the arc of a great circle bisecting the vertical angle of an isosceles spherical triangle is perpendicular to the base and bisects the base.

**Ex. 97.** Prove that the angles opposite the equal sides of an isosceles spherical triangle are equal.

## Proposition XXIII. Theorem

**704.** *If two spherical triangles on the same sphere, or on equal spheres, have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other,*

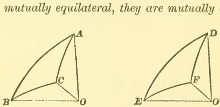
I. *They are congruent if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in opposite orders.*

The proof is left to the student.

## Proposition XXIV. Theorem

**705.** *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral, they are mutually equiangular.*




**Hypothesis.** $ABC$ and $DEF$ are mutually equilateral spherical ▲ on the equal sphere; $\overset{\frown}{BC}$ and $\overset{\frown}{EF}$ are homologous.

**Conclusion.** ▲ $ABC$ and $DEF$ are mutually equiangular.

*Suggestions.* — 1. Let $O$ and $O'$ be the centers of the respective spheres, and draw the radii to $A, B, C, D, E$, and $F$. Consider the two triedral angles $O\text{-}ABC$ and $O'\text{-}DEF$.

2. Compare the face angles of triedrals $O\text{-}ABC$ and $O'\text{-}DEF$.

3. Compare the diedral angles of $O\text{-}ABC$ and $O'\text{-}DEF$.

4. Now compare the ∠s $A$ and $D$; also the ∠s $B$ and $E$; also ∠s $C$ and $F$.

**Note.** — The theorem may be proved in a similar manner when the given spherical ▲ are on the same sphere.

**706. Cor.** *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral,*

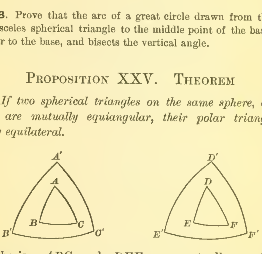
I. *They are congruent if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in opposite orders.*

**Ex. 98.** Prove that the arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

## Proposition XXV. Theorem

**707.** *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*




**Hypothesis.** $ABC$ and $DEF$ are mutually equiangular spherical $\triangle$ on the same sphere or equal spheres, $\angle A$ and $D$ being homologous; also, $\triangle A'B'C'$ is the polar $\triangle$ of $\triangle ABC$, and $\triangle D'E'F'$ of $\triangle DEF$, $A$ being the pole of $\widehat{B'C'}$, and $D$ of $\widehat{E'F'}$.

**Conclusion.** $\triangle A'B'C'$ and $D'E'F'$ are mutually equilateral.

*Suggestions.* — 1. Compare the measure of $\angle A$ and $\widehat{B'C'}$; of $\angle D$ and $\widehat{E'F'}$. Then compare $\widehat{B'C'}$ and $\widehat{E'F'}$.

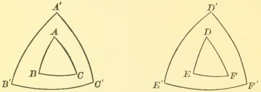
2. Proceed similarly for the other pairs of homologous sides.

**708. Cor.** *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

*Suggestion.* — Model the proof after that of § 707.

## Proposition XXVI. Theorem

**709.** *If two spherical triangles on the same sphere, or on equal spheres, are mutually equiangular, they are mutually equilateral.*




**Hypothesis.** $ABC$ and $DEF$ are mutually equiangular spherical ⨺ on the same sphere or equal spheres.

**Conclusion.** ⨺ $ABC$ and $DEF$ are mutually equilateral.

**Proof.** 1. Let $\triangle A'B'C'$ be the polar $\triangle$ of $ABC$, and $D'E'F'$ of $DEF$.

2. Since $\triangle ABC$ and $\triangle DEF$ are mutually equiangular, $\triangle A'B'C'$ and $\triangle D'E'F'$ are mutually equilateral. § 707

3. $\therefore \triangle A'B'C'$ and $\triangle D'E'F'$ are mutually equiangular.

4. $\therefore$ But $\triangle ABC$ is the polar $\triangle$ of $A'B'C'$ and $DEF$ of $D'E'F'$. § 655

5. $\therefore \triangle ABC$ and $\triangle DEF$ are mutually equilateral. Why?

**710. Cor.** *If two spherical triangles on the same sphere or on equal spheres are mutually equiangular,*

I. *They are congruent if the equal parts are arranged in the same order.*

II. *They are symmetrical if the equal parts are arranged in opposite orders.*

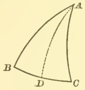
**Ex. 99.** Compare the theorem of Proposition XXVI with the corresponding theorem about two plane triangles.

**Ex. 100.** If three diameters of a sphere be drawn so that each is perpendicular to the other two, the planes determined by them divide the surface of the sphere into eight congruent tri-rectangular triangles.

**Note.** — Recall at this point all the theorems by which two spherical triangles can be proved congruent.

## Proposition XXVII. Theorem

**711.** *In an isosceles triangle, the angles opposite the equal sides are equal.*



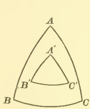
**Hypothesis.** In spherical triangle $ABC$, $\widehat{AB} = \widehat{AC}$.

**Conclusion.** $\angle B = \angle C$.

*Suggestion.* — Let $\widehat{AD}$, an arc of a great circle, bisect $\widehat{BC}$. Use § 705.

## Proposition XXVIII. Theorem

**712.** *If two angles of a spherical triangle are equal, the sides opposite are equal.*



**Hypothesis.** In spherical $\triangle ABC$, $\angle B = \angle C$.

**Conclusion.** $\widehat{AB} = \widehat{AC}$.

*Suggestions.* — 1. Let $\triangle A'B'C'$ be the polar $\triangle$ of $ABC$, $B$ being the pole of $\widehat{A'C'}$, and $C$ of $\widehat{A'B'}$.

2. Compare $\widehat{A'C'}$ and $\widehat{A'B'}$ by using § 656.

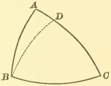
3. Compare $\angle B'$ and $\angle C'$, and from them determine the relation between $\widehat{AB}$ and $\widehat{AC}$.

**Ex. 101.** Prove that the great circle arcs drawn to the extremities of an arc of a great circle from any point on the great circle perpendicular to and bisecting it are equal.

**Ex. 102.** State and prove the converse of Ex. 101.

## Proposition XXIX. Theorem

**713.** *If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the side opposite the greater angle being the greater.*



**Hypothesis.** In spherical $\triangle ABC$, $\angle ABC > \angle C$.

**Conclusion.** $\widehat{AC} > \widehat{AB}$.

**Proof.** 1. Let $\widehat{BD}$, an arc of a great circle, meeting $\widehat{AC}$ at $D$, make $\angle CBD = \angle C$.

2. $\therefore \widehat{BD} = \widehat{DC}$. Why?

3. $\widehat{AD} + \widehat{BD} > \widehat{AB}$. Why?

4. $\therefore \widehat{AD} + \widehat{DC} > \widehat{AB}$.

5. $\therefore \widehat{AC} > \widehat{AB}$.

**714. Cor.** *If two sides of a spherical triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.*

An indirect proof based upon §§ 712 and 713 is quite easy.

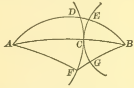
**Ex. 103.** What is the locus of points on the surface of a sphere which are equidistant from the extremities of an arc of a great circle of that sphere?

**Ex. 104.** Prove that the great circle arcs perpendicular to and bisecting the sides of a spherical triangle intersect in a point which is equidistant from the vertices of the triangle.

**Ex. 105.** Prove that a circle can be circumscribed about a spherical triangle.

## Proposition XXX. Theorem

**715.** *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semicircle which joins the two points.*



**Hypothesis.** Points $A$ and $B$ are on the surface of a sphere, and $\widehat{AB}$ is an arc of a great $\odot$, not greater than a semicircle.

**Conclusion.** $\widehat{AB}$ is the shortest line on the surface of the sphere between $A$ and $B$.

**Note.** — The following proof is divided into four parts, $(a)$, $(b)$, $(c)$, and $(d)$.

**Proof.** $(a)$ 1. Let $C$ be any point in $\widehat{AB}$.

2. Let $\widehat{DCF}$ and $\widehat{ECG}$ be arcs of small Ⓢ with $A$ and $B$ respectively as poles, and $\widehat{AC}$ and $\widehat{BC}$ as polar distances.

$(b)$ $\widehat{DCF}$ and $\widehat{ECG}$ have only point $C$ in common.

1. For let $F$ be any other point in $\widehat{DCF}$ and draw $\widehat{AF}$ and $\widehat{BF}$, arcs of great circles.

2. $\therefore \widehat{AF} = \widehat{AC}$. § 643

3. But $\widehat{AF} + \widehat{BF} > \widehat{AC} + \widehat{BC}$. Why?

4. Subtracting $\widehat{AF}$ from the first member of the inequality and its equal $\widehat{AC}$ from the second member,

$$\widehat{BF} > \widehat{BC}, \text{ or } \widehat{BF} > \widehat{BG}.$$

5. $\therefore F$ lies outside small $\odot ECG$, and $\widehat{DCF}$ and $\widehat{ECG}$ have only point $C$ in common.

$(c)$ The shortest line on the surface of the sphere from $A$ to $B$ must pass through $C$.

1. Let $ADEB$ be any line on the surface of the sphere between $A$ and $B$, not passing through $C$, and cutting $\widehat{DCF}$ and $\widehat{ECG}$ at $D$ and $E$ respectively.

2. Then, whatever the nature of line $AD$, it is evident that an equal line can be drawn from $A$ to $C$.

3. In like manner, whatever the nature of line $BE$, an equal line can be drawn from $B$ to $C$.

4. Hence a line can be drawn from $A$ to $B$ passing through $C$, equal to the sum of lines $AD$ and $BE$, and consequently less than $ADEB$ by the part $DE$.

5. Therefore no line which does not pass through $C$ can be the shortest line between $A$ and $B$.

(*d*) $\widehat{AB}$ is the shortest line from $A$ to $B$ on the surface of the sphere.

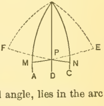
1. But $C$ is any point in $\widehat{AB}$.

2. Hence the shortest line from $A$ to $B$ must pass through every point of $AB$.

3. Then the great circle arc $AB$ is the shortest line on the surface of the sphere between $A$ and $B$.

**Ex. 118.** Prove that any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 573) from the sides of the angle.

(Prove $\widehat{PM} = \widehat{PN}$. Let $E$ be a pole of arc $AB$, and $F$ of arc $BC$. Spherical $\triangle$ $BPE$ and $BPF$ are symmetrical by § 702, II., and $\widehat{PE} = \widehat{PF}$.)




**Ex. 119.** Prove that a point on the surface of a sphere, equally distant from the sides of a spherical angle, lies in the arc of a great circle bisecting the angle.

(Fig. of Ex. 118. Prove $\angle ABP = \angle CBP$. Spherical $\triangle$ $BPE$ and $BPF$ are symmetrical by § 706.)

**Ex. 120.** What is the locus of points on the surface of a sphere equally distant from the sides of a spherical angle?

**Ex. 121.** Prove that the arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle.

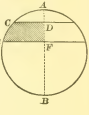
**Ex. 122.** Prove that a circle may be inscribed in any spherical triangle.

## Group C. Spherical Segments, Pyramids, and Wedges

**716.** A **Spherical Segment** is the portion of a sphere included between two parallel planes which intersect the sphere.

The portions of the planes bounding the segment are the *bases* of the segment; the perpendicular between the planes is the *altitude* of the segment.

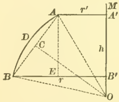
A spherical segment of one base is the spherical segment one of whose bounding planes is a tangent to the sphere.




If a semicircle $ACEB$ be revolved about diameter $AB$ as an axis, and $CD$ and $EF$ are perpendicular to $AB$, the portion of the plane bounded by $FECD$ generates a spherical segment whose altitude is $DF$, and whose bases have radii $CD$ and $EF$ respectively; the portion $ACD$ generates a spherical segment of one base whose altitude is $AD$.

**717.** *If $r$ and $r'$ are the radii of the bases, $h$ the altitude, and $v$ the volume of a spherical segment, then*

$$v = \frac{1}{2}\pi(r^2 + r'^2)h + \frac{1}{6}\pi h^3.$$




Let $O$ be the center of $\widehat{ADB}$; let $AA'$ and $BB'$ be $\perp$ to the diameter $OM$; let $AA' = r'$, $BB' = r$, $A'B' = h$. Let the whole figure revolve about $OM$ as an axis. Let $v =$ the volume of the resulting spherical segment.

**Solution.** 1. Draw $OA$, $OB$, and $AB$; draw $OC \perp AB$, and $AE \perp BB'$. Let $OA = R$.

2. Now, vol. $ADBB'A' =$ vol. $ACBD +$ vol. $ABB'A'$. (1)
3. Also, vol. $ACBD =$ vol. $OADB -$ vol. $OAB$.
4. vol. $OADB = \frac{2}{3}\pi R^2 h$. § 694
5. And, vol. $OAB =$ area $AB \times \frac{1}{3} OC$ § 691
6. $= h \times 2\pi OC \times \frac{1}{3} OC$ § 668
7. $= \frac{2}{3}\pi \overline{OC}^2 h$.
8. $\therefore$ vol. $ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$
9. $= \frac{2}{3}\pi(R^2 - \overline{OC}^2)h$.
10. But, $R^2 - \overline{OC}^2 = \overline{AC}^2$ Why?
11. $= (\frac{1}{2} AB)^2$
12. $= \frac{1}{4}\overline{AB}^2$.
13. $\therefore$ vol. $ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h$.
14. Now, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2$
15. $= (r - r')^2 + h^2$.
16. $\therefore$ vol. $ACDB = \frac{1}{6}\pi[(r - r')^2 + h^2]h$.
17. Also, vol. $ABB'A' = \frac{1}{3}\pi(r^2 + r'^2 + rr')h$. § 621
18. Substituting in step 2, vol. $ADBB'A'$
$= \frac{1}{6}\pi[(r - r')^2 + h^2]h + \frac{1}{6}\pi(2r^2 + 2r'^2 + 2rr')h$
19. $= \frac{1}{6}\pi(r^2 - 2rr' + r'^2 + h^2 + 2r^2 + 2r'^2 + 2rr')h$
20. $= \frac{1}{6}\pi(3r^2 + 3r'^2)h + \frac{1}{6}\pi h^3$
21. $= \frac{1}{2}\pi(r^2 + r'^2)h + \frac{1}{6}\pi h^3$.

**Ex. 123.** Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

**718.** A **Spherical Wedge** is a solid bounded by a lune and the planes of its bounding arcs.

Evidently, two spherical wedges in the same sphere, or equal spheres, are congruent when their angles are equal.

Also, it is evident that two wedges in the same sphere can be added by placing them so that they have one common bounding plane. The angle of the sum is equal to the sum of the angles of the wedges.

**Note.** — Review at this time § 675 and § 676, noting the analogy between wedges and lunes.

**719.** It can be proved as in § 677 that two wedges have the same ratio as their angles. (Cf. § 677.)

**720.** A sphere may be regarded as a wedge whose angle is 360°. (Cf. § 678.)

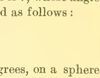
Therefore, a wedge of a sphere whose radius is $r$, whose angle contains $A$ degrees has a volume $v$ determined as follows:

$$\frac{v}{\frac{4}{3}\pi r^3} = \frac{A}{360}, \text{ or } v = \frac{\pi r^3 A}{270}.$$

**721.** Since the lune whose angle is $A$ degrees, on a sphere whose radius is $r$, has its area expressed by the formula $\frac{\pi r^2 A}{90}$, (§ 680)

$\therefore$ the volume of a wedge equals one third the radius of the sphere multiplied by the area of the lune which forms its base.

**722.** A **Spherical Pyramid** is a solid bounded by the spherical polygon and the planes of its sides; as $O\text{-}ABCD$ in the adjoining figure.

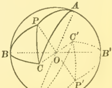



The center of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base*.

Two spherical pyramids are congruent when their bases are congruent, for they can be made to coincide.

**723.** Two spherical pyramids whose bases are symmetrical isosceles spherical triangles are congruent, for their bases are congruent by § 683.

**724.** *Two spherical pyramids corresponding to a pair of vertical triedral angles are equal.* (Cf. § 685.)




*Suggestions.* — 1. Recall the proof of § 685.

2. Compare spherical pyramids $O\text{-}APB$, $O\text{-}BPC$, and $O\text{-}CPA$ with spherical pyramids $O\text{-}A'P'B'$, $O\text{-}B'P'C'$, and $O\text{-}C'P'A'$, respectively.

**725.** *The volume of a triangular spherical pyramid equals one half the volume of a spherical wedge whose angle is the spherical excess of the base of the pyramid.*

The proof is exactly like that for § 687.

**726.** *If the radius of the sphere is $r$ and the excess of the base of a triangular spherical pyramid is $E$, and the volume of the spherical pyramid is $v$, then*

$$v = \frac{1}{2}\frac{\pi r^3 \cdot E}{270} = \frac{\pi r^3 E}{540}. \qquad \S\ 720$$

**727.** The same formula may be employed to find the volume of any spherical pyramid, with the understanding that $E$ is the spherical excess of the base of the pyramid, measured in degrees.

**728.** In the case of any spherical pyramid, the area of the base is $\frac{\pi r^2 E}{180}$ (§ 689). Hence the volume of any spherical pyramid is one third the area of its base multiplied by the radius of the sphere.

**Ex. 124.** Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°, if the volume of the sphere is 192.

**Ex. 125.** Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°, if the diameter of the sphere is 12.

**Ex. 126.** The volume of a triangular spherical pyramid, the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$. What is the volume of the sphere?

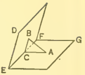
**Ex. 127.** Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°, if the volume of the sphere is 180.

**Ex. 128.** The volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146° is $12\frac{3}{4}$. What is the volume of the sphere?

**Ex. 129.** What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

## SUPPLEMENTARY EXERCISES

**Ex. 1.** Two planes $DEF$ and $GEF$ intersect in line $EF$. $A$ is any point in plane $GEF$. If $AC$ be drawn perpendicular to $EF$, and $AB$ perpendicular to plane $DEF$, prove the plane determined by $AC$ and $BC$ perpendicular to $EF$.




**Ex. 2.** Prove that the line joining the mid-points of one pair of opposite sides of a quadrilateral in space bisects the line joining the mid-points of the other pair of sides.

**Ex. 3.** If two intersecting planes pass through two parallel lines, their intersection is parallel to the parallel lines.

**Ex. 4.** If three planes intersect in pairs, the lines of intersection are either parallel or concurrent.

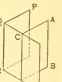
*Suggestions.* — Case (*a*) 1. Assume that two lines of intersection meet at a point $P$. 2. Prove that $P$ is on the third line of intersection. Case (*b*) 1. Assume that two lines of intersection are parallel. 2. Prove that the third line of intersection is parallel to the other two by an indirect proof.

**Ex. 5.** Prove that a line parallel to each of two intersecting planes is parallel to their intersection.

**Hyp.** $AB$ is $\parallel$ planes $PR$ and $QS$.

**Con.** $AB \parallel QR$.

*Suggestion.* — Pass a plane through $AB \parallel PR$; then use § 471.

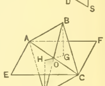



**Ex. 6.** If a plane be drawn through a diagonal of a parallelogram the perpendiculars to it from the extremities of the other diagonal are equal.

**Hyp.** $ABCD$ is a $\square$.

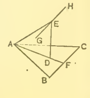
$BG$ and $DH \perp$ plane $AECF$.

**Con.** $BG = DH$.




**Ex. 7.** $D$ is any point in perpendicular $AF$ from $A$ to side $BC$ of triangle $ABC$. If line $DE$ be drawn perpendicular to the plane of $ABC$, and line $GH$ be drawn through $E$ parallel to $BC$, prove line $AE$ perpendicular to $GH$.

*Suggestion.* — Prove $BC \perp$ to plane $AED$ and then $GH \perp$ plane $AED$.

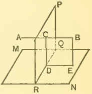


**Ex. 8.** (*a*) Through a line which is parallel to a plane, a plane can be drawn parallel to the given plane.

(*b*) Is the construction possible if the given line is not parallel to the given plane?

**Ex. 9.** If a plane, parallel to the edge of a diedral angle, intersects the faces of the angle, its intersections with the faces are parallel to the edge and to each other.

**Ex. 10.** If a straight line and a plane are both perpendicular to a given plane, they are parallel, unless the line lies in the plane.




**Hyp.** $CB \perp$ plane $PR$; plane $MN \perp$ plane $PR$.

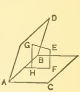
**Con.** $CB \parallel MN$.

*Suggestions.* — 1. Draw $CD \perp RQ$, and let the plane determined by $CB$ and $CD$ meet $MN$ in $DE$.

2. Prove $CB \parallel DE$ and hence $\parallel$ plane $MN$ by § 466.

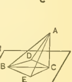
**Ex. 11.** If a plane is perpendicular to one of two perpendicular planes, its intersection with the other plane is also perpendicular to the first plane.

**Ex. 12.** From any point $E$ within diedral angle $CABD$, $EF$ and $EG$ are drawn perpendicular to faces $ABC$ and $ABD$, respectively, and $GH$ perpendicular to face $ABC$ at $H$. Prove $FH$ is perpendicular to $AB$.




*Suggestion* — Prove that $FH$ lies in the plane of $EF$ and $EG$, by § 498; also consider the relation of $AB$ and plane $GEF$.

**Ex. 13.** If $BC$ is the projection of line $AB$ upon plane $MN$, and $BD$ and $BE$ be drawn in the plane making $\angle CBD = \angle CBE$, prove $\angle ABD = \angle ABE$.




*Suggestions.* — 1. Lay off $BD = BE$, and draw lines $AD$, $AE$, $CD$, and $CE$.

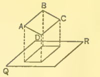
2. Prove $\triangle ABD$ and $ABE$ congruent.

**Ex. 14.** If a line is perpendicular to one of two intersecting planes, its projection upon the other is perpendicular to the intersection of the two planes.

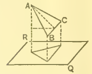
*Suggestion.* — If $EG \perp$ plane $AD$, and $FH$ is the projection of $EG$ on plane $BC$, prove $FH \perp AB$.

**Ex. 15.** If a straight line intersects two parallel planes, it makes equal angles with them.

**Ex. 16.** The base of a rectangle $ABCD$ is 10, and its altitude 8. Side 10 is parallel to plane $QR$. Side 8 makes an angle of 60° with $QR$. Find the area of the projection of $\square$ $ABCD$ on plane $QR$, correct to three decimal places.




**Ex. 17.** An equilateral $\triangle ABC$, whose area is 25, has its side $BC$ parallel to a plane $QR$. The plane of $\triangle ABC$ makes an angle of 45° with plane $QR$. Find the area of the projection of $\triangle ABC$ on plane $RQ$.



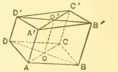

**Ex. 18.** Find the lateral area of a regular triangular prism each side of whose base is 5 and whose altitude is 8.

**Ex. 19.** Prove that the upper base of a truncated parallelopiped is a parallelogram.

**Ex. 20.** Prove that the sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

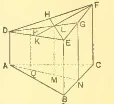
*Suggestions.* — 1. What kind of figure is $AA'C'C$?

2. Compare $AA' + CC'$ with $OO'$.



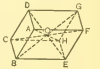

**Ex. 21.** Prove that the perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one third the sum of the lateral edges.

*Suggestion.* — Let $P$ be the mid-point of $DL$, and draw $PQ \perp ABC$; express $LM$ in terms of $PQ$ and $GN$.




**Ex. 22.** Prove that the sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

*Suggestion.* — Recall Ex. 142, Book III, p. 184.




**Ex. 23.** Determine the approximate area of the base of a bin 6 ft. deep that will hold 250 bu. of grain. (One bu. = 2150.42 cu. in.)

**Ex. 24.** Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

**Ex. 25.** Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

**Ex. 26.** The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

*Suggestion.* — Represent the length of the edge by $x$.

**Ex. 27.** Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

*Suggestion.* — Represent the dimensions of the base by $x$ and $y$.

**Ex. 28.** Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

**Ex. 29.** A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain?

**Ex. 30.** Prove that the volume of any oblique prism is equal to the product of the area of a right section by the length of a lateral edge.

**Ex. 31.** Prove that the volume of a regular prism is equal to its lateral area multiplied by one half the apothem of the base.

**Ex. 32.** The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21, respectively. Find its lateral area.

**Ex. 33.** The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

*Suggestion.* — Represent the altitude by $x$, and the side of the base by $y$.

**Ex. 34.** The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in., respectively. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

**Ex. 35.** At what distance from the altitude must a plane parallel to the base be drawn so that the area of the section will be one half the base?

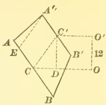
**Ex. 36.** In Ex. 35, replace the fraction $\frac{1}{2}$ by the fraction $\frac{1}{3}$ and solve the resulting exercise.

**Ex. 37.** In Ex. 35, replace the fraction $\frac{1}{2}$ by the fraction $\frac{2}{3}$ and solve the resulting exercise.

**Ex. 38.** Prove that the volume of a regular-pyramid is equal to its lateral area, multiplied by one third the distance from the center of its base to any lateral face.

*Suggestion.* — Pass planes through the lateral edges and the center of the base.

**Ex. 39.** Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, respectively, and whose altitude is 12.




*Suggestion.* — Let $ABB'A'$ be a lateral face of the frustum, and $O$ and $O'$ the centers of the bases; draw lines $OC \perp AB$, $O'C' \perp A'B'$, $C'D \perp OC$, and $A'E \perp AB$; also lines $OO'$ and $CC'$.

**Ex. 40.** The bases of a frustum of a pyramid are rectangles, whose sides are 27 and 15, and 9 and 5, respectively, and the line joining their centers is perpendicular to each base. If the altitude of the frustum is 12, find its lateral area and volume.

**Ex. 41.** Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, respectively, and whose lateral edge is 5.

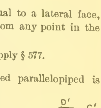
**Ex. 42.** The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.

**Ex. 43.** Find the volume of the rectangular prismoid the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

**Ex. 44.** The volume of a triangular prism is equal to a lateral face, multiplied by one half its perpendicular distance from any point in the opposite lateral edge.

*Suggestion.* — Draw a rt. section of the prism, and apply § 577.

**Ex. 45.** Prove that the volume of a truncated parallelopiped is equal to the area of a right section multiplied by one fourth the sum of the lateral edges.




**Ex. 46.** Prove that a plane passed through the center of a parallelopiped divides it into two equal solids.

**Ex. 47.** The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by the distance between the centers of the bases.

*Suggestion.* — By Ex. 45, the distance between the centers of the bases may be proved equal to one fourth the sum of the lateral edges.

**Ex. 48.** How many square feet of heating surface are there in a hot-water conducting pipe 9 feet long and 2 inches in outside diameter?

**Ex. 49.** Determine the lateral area of the right circular cylinder formed by revolving a rectangle, having base $b$ and altitude $h$,

(*a*) about its base; (*b*) about its altitude.

**Ex. 50.** The lateral area of a cylinder of revolution is $120\pi$. The area of the base is $36\pi$. Find the altitude.

**Ex. 51.** The cross section of a tunnel, $2\frac{1}{2}$ mi. in length, is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle; how many cubic yards of material were taken out in its construction? ($\pi = 3.1416$.)

**Ex. 52.** What must be the length in inches of a 10-gal. gasoline tank which is 10 in. in diameter?

**Ex. 53.** Determine the volume generated when a rectangle of base $b$ and altitude $h$

(*a*) revolves about its side $b$; (*b*) revolves about its side $h$.

**Ex. 54.** Two right circular cylinders have equal altitudes, but the radius of the base of the one is double the radius of the base of the other. Compare (*a*) their lateral areas; (*b*) their volumes.

**Ex. 55.** A regular hexagonal prism is inscribed in a right circular cylinder whose altitude is 10 in. and the radius of whose base is 3 in. Determine the difference between the volumes of the prism and cylinder.

**Ex. 56.** Prove that the volume of a cylinder of revolution is equal to its lateral area multiplied by one half the radius of its base.

**Ex. 57.** Express by a formula the volume of a round cast-iron column of length $l$ ft., thickness $t$ in., and outside diameter $d$ in.

**Ex. 58.** Given the radius of the base $R$ and the total area $T$ of a cylinder of revolution, find its volume.

(Find $H$ from the equation $T = 2\pi RH + 2\pi R^2$.)

**Ex. 59.** Given the diameter of the base $D$ and the volume $V$ of a cylinder of revolution, find its lateral area and total area.

**Ex. 60.** The volume of a circular cone is $V$. What is the effect upon the volume:

(*a*) if the radius of the base is doubled?

(*b*) if the altitude is doubled?

(*c*) if both the radius and the altitude are doubled?

**Ex. 61.** The altitude of a cone of revolution is 27 in., and the radius of its base is 16 in. What is the diameter of the base of an equal cylinder, whose altitude is 16 in.?

**Ex. 62.** A plane is passed parallel to the base of a circular cone so as to bisect the altitude. What is the ratio of the two parts into which the given cone is divided?

**Ex. 63.** Determine the lateral area of a right circular cone whose volume is $320\pi$ cu. in., and whose altitude is 15 in.

**Ex. 64.** Determine the volume of a cone of revolution whose slant height is 29 in., and whose lateral area is $580\pi$ sq. in.

**Ex. 65.** If the altitude of a cone of revolution is three fourths the radius of its base, the volume is equal to its lateral area multiplied by one fifth the radius of its base.

**Ex. 66.** Given the altitude $H$ and the volume $V$ of a right circular cone. Derive the formula for the lateral area in terms of $V$ and $H$.

**Ex. 67.** Given the slant height $L$ and the lateral area $S$ of a right circular cone. Derive the formula for its volume in terms of $S$ and $L$.

**Ex. 68.** Find the lateral area of the frustum of a right circular cone, whose altitude is 8 in., if the radii of its bases are 6 in. and 3 in., respectively.

**Ex. 69.** A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end and 8 in. in diameter at the other. How many cubic inches of metal are there in it?

**Ex. 70.** Prove that a frustum of a circular cone is equal to three cones whose common altitude is the altitude of the frustum, and whose bases equal the lower base, the upper base, and the mean proportional between the bases of the frustum.

**Ex. 71.** The area of the entire surface of a frustum of a cone of revolution is $306\pi$ sq. in., and the radii of its bases are 11 in. and 5 in., respectively. Find the lateral area and the volume of it.

**Ex. 72.** The volume of a frustum of a right circular cone is $6020\pi$ cu. in., its altitude is 60 in., and the radius of its lower base is 15 in. Find the radius of the upper base and its lateral area.

**Ex. 73.** Find the diameter and the area of the surface of a sphere whose volume is $\frac{1125}{2}\pi$ cu. in.

**Ex. 74.** The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3; what is the diameter of an equal sphere?

**Ex. 75.** Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

**Ex. 76.** A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{3}{8}$ and $\frac{9}{32}$ the surface and volume of the sphere.

**Ex. 77.** Given the area of the surface of a sphere $S$ to find its volume.

**Ex. 78.** Given the volume of a sphere $V$ to find the area of its surface.

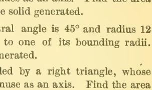
**Ex. 79.** A portion of a plane bounded by an equilateral triangle, whose side is 6, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume of the solid generated.

**Ex. 80.** A circular sector whose central angle is 45° and radius 12 revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

**Ex. 81.** A portion of a plane bounded by a right triangle, whose legs are $a$ and $b$, revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume of the solid generated.

**Ex. 82.** A portion of a plane bounded by an equilateral triangle, whose altitude is $h$, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume of the solid generated.

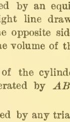
**Ex. 83.** A portion of a plane bounded by an equilateral triangle, whose side is $a$, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume of the solid generated.

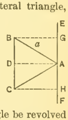



(The solid generated is the difference of the cylinder generated by $BCHG$, and the cones generated by $ABG$ and $ACH$.)

**Ex. 84.** If a portion of a plane bounded by any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one third the altitude.

**Ex. 85.** If a portion of a plane bounded by any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one third the altitude.




**Ex. 86.** Find the volume of a spherical sector, the altitude of whose base is 12, the diameter of the sphere being 25.

## IMPORTANT DEFINITIONS AND THEOREMS OF PLANE GEOMETRY

**§ 8.** (*a*) One and only one straight line can be drawn through two points.

(*b*) A straight line can be extended indefinitely in each direction.

**§ 11.** Two straight lines can intersect at only one point.

**§ 14.** The straight line segment is the shortest line between two points.

**§ 16.** A circle is a closed curved line (in a plane) all points of which are equidistant from a point within called the center.

**§ 17.** All radii of the same circle or of equal circles are equal.

**§ 20.** An angle is the figure formed by two rays drawn from the same point.

**§ 24.** Adjacent angles are two angles that have a common vertex and a common side between them.

**§ 26.** If one straight line meets another straight line so that the adjacent angles formed are equal, each of these angles is a right angle.

**§ 27.** All right angles are equal.

**§ 29.** Two lines are perpendicular if they form a right angle.

**§ 34.** The sum of all the successive adjacent angles around a point on one side of a straight line is one straight angle.

**§ 35.** The sum of all the successive adjacent angles around a point is two straight angles.

**§ 36.** Two angles are complementary if their sum equals a right angle.

**§ 37.** Complements of the same angle or of equal angles are equal.

**§ 38.** Two angles are supplementary if their sum is equal to a straight angle.

**§ 39.** If two adjacent angles have their exterior sides in a straight line, they are supplementary.

**§ 40.** If two adjacent angles are supplementary, their exterior sides are in a straight line.

**§ 41.** Supplements of the same angle or of equal angles are equal.

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